The Effective Mass of an Oscillating Spring

J. G. Fox, and J. Mahanty

Citation: American Journal of Physics **38**, 98 (1970); doi: 10.1119/1.1976240
View online: https://doi.org/10.1119/1.1976240
View Table of Contents: http://aapt.scitation.org/toc/ajp/38/1
Published by the American Association of Physics Teachers

Articles you may be interested in

Effective Mass of an Oscillating Spring
The Physics Teacher **45**, 100 (2007); 10.1119/1.2432087

The Oscillating Spring and Weight—An Experiment Often Misinterpreted
American Journal of Physics **37**, 447 (1969); 10.1119/1.1975602

Finding the Effective Mass and Spring Constant of a Force Probe from Simple Harmonic Motion
The Physics Teacher **54**, 138 (2016); 10.1119/1.4942130

Influence of the mass of the spring on its static and dynamic effects
American Journal of Physics **47**, 1076 (1979); 10.1119/1.11978

A Demonstration of the Spring-Mass Correction
American Journal of Physics **37**, 645 (1969); 10.1119/1.1975733

Equivalent mass of a coil spring
The Physics Teacher **38**, 140 (2000); 10.1119/1.880475

ACKNOWLEDGMENTS

A portion of the equipment used in these experiments was purchased with funds from a Ford Foundation grant. The authors gratefully acknowledge this support. They wish also to thank the Rockefeller Foundation for its support and encouragement of this work.

* Special Staff Member of the Rockefeller Foundation, 1968-69. On leave from Amherst College, Amherst, Mass.

[1] W. C. Michels, *Electrical Measurements and Their Application* (D. Van Nostrand Co., Inc., Princeton, N. J., 1957), pp. 232-235.

[2] T. B. Brown, Ed., R. H. Bacon, S. C. Brown, R. H. Howe, R. R. Palmer, L. R. Weber, R. L. Weber, and F. T. Worrell, *The Lloyd William Taylor Manual of Advanced Undergraduate Experiments in Physics* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1959), pp. 324–327.

[3] H. V. Malmstadt, C. G. Enke, and E. C. Toren, *Electronics for Scientists* (W. A. Benjamin, Inc., New York, 1963), pp. 356–358, 351–352.

[4] Analog Devices, Inc., 221 5th Street, Cambridge, Mass. Price of the model 150, plus mating socket, is \$32.75. The power supply consists of two Mallory TR 132R mercury cells (\$2.34/pair). As the current drain on batteries is 1 mA, the operating lifetime of the batteries should be approximately 1000 h.

AMERICAN JOURNAL OF PHYSICS VOLUME 38, NUMBER 1 JANUARY 1970

The Effective Mass of an Oscillating Spring

J. G. Fox* and J. Mahanty

Department of Physics, Indian Institute of Technology, Kanpur, India

(Received 30 December 1968; revision received 14 July 1969)

We have calculated the effective mass of the spring in an oscillating mass–spring system. It varies from $4/\pi^2$ to $\frac{1}{3}$ the mass of the spring as the suspended mass varies from 0 to ∞. This and the various predicted modes can be easily verified experimentally.

A mass M attached to a spring having a force constant K is a system well-known to every student of physics. It epitomizes simple harmonic motion. The spring is usually assumed to be massless. When the motion is studied in more detail, the student is led to believe that $\frac{1}{3}$ of the mass m of the spring, the effective mass, should be added to M when calculating the frequency. Derivations of the effective mass are given in some textbooks,[1] but the result is not always $0.33m$. Sometimes it is $0.50m$.[2] As a matter of experimental fact, the result is neither of these values but something in between. For a system with $M=0$, as in the case of an oscillating "slinky" with no mass attached, the effective mass is $0.40m$.[3] These discrepancies suggest a more detailed examination of this familiar old system: it has perhaps been treated in too cavalier a fashion. We give below an analysis of the longitudinal oscillations of a mass M attached to a spring having mass m and force constant K. We first solve the problem for the gravity-free case, and then we include the effect of gravity on a vertical mass–spring system.

Note Added in Proof: After this manuscript was submitted, two other discussions of this problem have appeared: H. L. Armstrong, Amer. J. Phys. **37,** 447 (1969); F. W. Sears, Amer. J. Phys. **37,** 645 (1969).

I. GRAVITY OFF

If the coordinates of the fixed and free ends of the spring are $x=0$ and $x=x_0$, respectively, then the linear density is $\sigma=m/x_0$ and the force constant of a unit length is Kx_0. The equation of motion for $U(x,\ t)$, the displacement from equilibrium of a point at x is

$$\frac{\partial^2 U(x,t)}{\partial x^2}-\left(\frac{\sigma}{Kx_0}\right)\frac{\partial^2 U(x,t)}{\partial t^2}=0. \qquad (1)$$

Separation of variables with the usual sinusoidal time variation of U at frequency ω gives, for the amplitude $u(x)$, the equation

$$(d^2u/dx^2)+(\omega^2\sigma/Kx_0)u=0. \qquad (2)$$

One of the boundary conditions is $u(x)\,|_{x=0}=0$.

Hence the solution of Eq. (2) is

$$u(x) = A \sin[\omega x(\sigma/Kx_0)^{1/2}]. \quad (3)$$

The other condition is

$$T(x_0) = Kx_0(\partial U/\partial x)\,|_{x=x_0} = -M(\partial^2 U/\partial t^2)\,|_{x=x_0},$$

where T is the tension. Substitution of the derivatives of $U(x, t) = A \sin\omega t \sin[\omega x(\sigma/Kx_0)^{1/2}]$ yields immediately

$$\mu \tan\mu = m/M \quad (4)$$

where $\mu = \omega(m/K)^{1/2}$. We postpone discussion of the solutions of this equation until the latter part of Sec. II.

II. GRAVITY ON

When the same mass–spring system is vertically supported, an element dx of the unstretched spring with its force constant Kx_0/dx is stretched to dx' in the static case under the influence of gravity, and by an amount dU in the dynamic case, so that

$$T(x) = Kx_0(\partial U/\partial x + dx'/dx - 1). \quad (5)$$

At this point we can proceed to set up the differential equation and solve it using as independent variables either x and t or x' and t. Physically, we expect that in both cases the same frequency spectrum will be obtained. In terms of x, the linear density of the spring is constant and the problem is easier to solve. In terms of x', the nonuniform density makes the solution more complicated, but the amplitude $u(x')$ is more directly related to observation. Here the problem is done in terms of x, and the solution in terms of x' appears in the Appendix sketch.

From Eq. (5) the net force on the segment dx is given by $dT = Kx_0(\partial^2 U/\partial x^2 + d^2x'/dx^2)\,dx$. The equation of motion is obtained by setting this equal to $[(\sigma dx)(\partial^2 U/\partial t^2) - \sigma g dx]$, i.e.,

$$\partial^2 U/\partial x^2 + d^2x'/dx^2 = (\sigma/Kx_0)(\partial^2 U/\partial t^2) - \sigma g/Kx_0. \quad (6)$$

To obtain the relationship between x and x', we use the condition of static equilibrium of the length of the spring between x and x_0,

$$(Kx_0/dx)(dx' - dx) = \sigma g(x_0 - x) + Mg. \quad (7)$$

Hence,

$$dx'/dx = (Mg + mg + Kx_0 - \sigma g x)/Kx_0,$$

and

$$d^2x'/dx^2 = -\sigma g/Kx_0 \quad (8)$$

Substitution of Eq. (8) in Eq. (6) results in the former equation of motion (1). The first boundary condition, $u(x)\,|_{x=0} = 0$, is the same as before. The second is obtained by using Eq. (5) to get the tension at $x = x_0$ and equating it to $[Mg - M(\partial^2 U/\partial t^2)]$. With the help of Eq. (7), the second boundary condition then becomes

$$Kx_0(\partial U/\partial x)\,|_{x=x_0} = -M(\partial^2 U/\partial t^2)\,|_{x=x_0},$$

which is the same as before. Since the equation of motion and the boundary conditions are the same as those leading to the solution given in Eq. (4), the latter is also the solution for this case, and the frequencies and the amplitudes expressed as functions of the coordinate x are identical in the presence and absence of gravity. The same is true for the effective mass.

When $M \gg m$ the first root of Eq. (4) is given by

$$\mu(\mu + \mu^2/3 + \cdots) \cong m/M, \qquad \omega \cong [K/(M + m/3)]^{1/2}$$

and the other roots are given by

$$\mu \cong n\pi, \qquad \omega \cong \pi n(K/m)^{1/2}, \qquad n = 1, 2, 3, \cdots.$$

At the other extreme, when $M = 0$, we obtain

$$(\cot\mu)/\mu = 0, \text{ or } \mu = (2n+1)\pi/2,$$

$$\omega = (2n+1)[K/(4m/\pi^2)]^{1/2},$$

$$n = 0, 1, 2, \cdots.$$

Thus in the fundamental mode, as $M/m \to \infty$, the effective mass of the spring approaches $m/3$, while for $M = 0$ it is $4m/\pi^2 = 0.405m$. In the higher modes the concept loses its usefulness. For intermediate values of m/M the tabulated solutions of Eq. (4)[4] can be used to calculate and plot the effective mass for the fundamental mode or the computer can be used as was done for Fig. 1.

The first several modes of oscillation given by the roots of Eq. (4) can be readily demonstrated. We have excited the modes in a spring hung vertically from a support that was driven by an electromagnet connected to the amplified output of an audio-oscillator. When $M = 0$, modes of oscillation with 0, 1, 2, 3, $\cdots$ nodes were readily observed for which the frequencies were in the ratios 1, 3, 5, 9, $\cdots$. When M is equal to

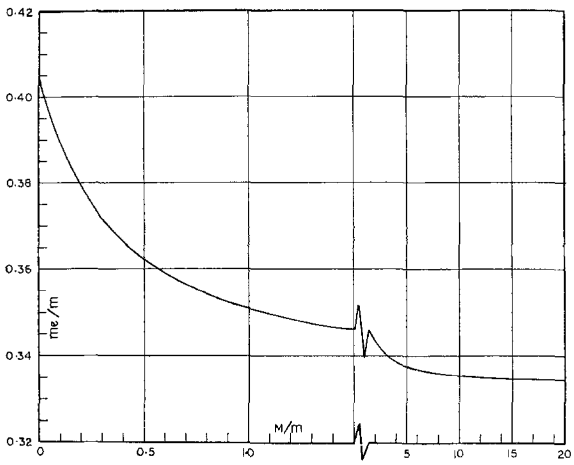

FIG. 1. The effective mass, m_e, of a spring having a mass, m, with an added mass, M.

several times m, the first, second, third, etc. harmonics consisted of vibrations with nodes approximately at the ends of the spring and with 0, 1, 2, ··· additional nodes in between, and frequencies in the approximate ratios 1, 2, 3, ···. As mentioned in the opening paragraph, it is not difficult to discover in the elementary physics laboratory that, for $M=0$, the effective mass of the spring is 0.40 times its actual mass. The whole problem makes an interesting exercise for the undergraduate laboratory and/or classroom mechanics course demonstrating somewhat unanticipated aspects of a simple, familiar system.

APPENDIX

The solution in terms of x', mentioned in connection with Eq. (5), is obtained as follows. From Eq. (5) dT/dx' is expressed in terms of derivatives of U and x with respect to x'. x is then eliminated by means of Eq. (8) and its integral to yield the equation

$$\frac{d^2u}{dz^2}+\frac{(1/2z)du}{dz}+\frac{\eta^2u}{z}=0,$$

where $z=[(Mg+mg+Kx_0)^2/2mgK]-x'$ and $\eta^2=\omega^2/2g$. The solutions of this equation are $Az^{1/2}j_0(2\eta z^{1/2})$ and $Bz^{1/2}y_0(2\eta z^{1/2})$, where j_0 and y_0 are the spherical Bessel and Neumann functions. The two boundary conditions and the requirement of a nontrivial solution for the arbitrary constants A and B lead to Eq. (4).

A simple but pedagogically useful example of the use of the δ function is provided by treating M not as a boundary condition but as an integral part of the system, i.e., as a term $M\delta(x-x_0)$ to be added to the density σ in the equation of motion [Eq. (2)]. Then the functions $\Phi_n(x)=(2/x_0)^{1/2}\sin[(2n+1)\pi x/2x_0]$, complete and orthonormal for $0<x<x_0$, can be used in the equation of motion to expand $u(x)$ and, through the closure relation, $\delta(x)$. The expansion coefficients can then be evaluated, x set equal to x_0 in $u(x)$, and the partial fraction expansion of the cotangent[5] used to obtain Eq. (4) again.

* Permanent address: Department of Physics, Carnegie-Mellon University, Pittsburgh, Pa. 15213.

[1] D. E. Christie, *Intermediate College Mechanics* (McGraw-Hill Book Co., New York, 1952), p. 290.

[2] J. E. Younger, *Advanced Dynamics* (The Ronald Press Co., New York, 1958), p. 132.

[3] Results of measurements by the authors.

[4] *Handbook of Mathematical Functions*, M. Abramowitz and I. A. Stegun, Eds. (Natl. Bur. Std., Appl. Math. Ser. 55) (U. S. Department of Commerce, Washington, D. C., 1964), p. 362.

[5] See Ref. 4, p. 75.