

Type theory and category theory

Michael Shulman

<http://www.math.ucsd.edu/~mshulman/hottminicourse2012/>

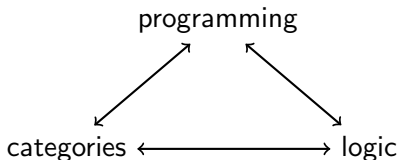
10 April 2012

The three faces of type theory

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- ② A foundation for mathematics.
- ③ A calculus for category theory.

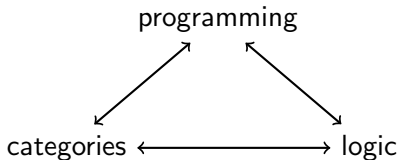
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- ① + ② : A computable foundation for mathematics.
 - ② + ③ : A way to internalize mathematics in categories.
 - ① + ③ : A categorical description of programming semantics.



The three faces of homotopy type theory

- 1 A programming language.
 - 2 A foundation for mathematics based on homotopy theory.
 - 3 A calculus for $(\infty, 1)$ -category theory.
- 1 + 2 : A computable foundation for homotopical mathematics.
- 2 + 3 : A way to internalize homotopical mathematics in categories.
- 1 + 3 : A categorical description of programming semantics.



Minicourse plan

- **Today:** Type theory, logic, and category theory
- **Wednesday:** Homotopy theory in type theory
- **Thursday:** Type theory in $(\infty, 1)$ -categories
- **Friday:** Current frontiers

- 1 Type theory and category theory
- 2 Type constructors and universal properties
- 3 Type theory and logic
- 4 Predicate logic and dependent types
- 5 Equality types

Type theory consists of rules for manipulating **typing judgments**:

$$(x_1 : A_1), (x_2 : A_2), \dots, (x_n : A_n) \vdash (b : B)$$

- The x_i are variables, while b stands for an arbitrary expression.
- The turnstile \vdash and commas are the “outermost” structure.

This should be read as:

In the context of variables x_1 of type A_1 , x_2 of type A_2 , \dots ,
and x_n of type A_n , the expression b has type B .

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The meanings of a typing judgment

$$(x_1 : A_1), (x_2 : A_2), \dots (x_n : A_n) \vdash (b : B)$$

- 1 **Programming:** A_i, B are datatypes (`int`, `float`, ...); b is an expression of type B involving variables x_i of types A_i .
- 2 **Foundations:** A_i, B are “sets”, b specifies a way to construct an element of B given elements x_i of A_i .
- 3 **Category theory:** A_i, B are objects, b specifies a way to construct a morphism $\prod_i A_i \rightarrow B$.

Type constructors

The rules of type theory come in packages called **type constructors**. Each package consists of:

- 1 **Formation**: a way to construct new types.
- 2 **Introduction**: ways to construct terms of these types.
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- 4 **Computation**: what happens when we follow 2 by 3 .

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- 4 $(\lambda x.b)(a)$ computes to b with a substituted for x .

square := $\lambda x.(x * x)$

```
int square(int x) { return (x * x); }
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def square(x):  
  return (x * x)
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square :: Int -> Int  
square x = x * x
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fun square (n:int):int = n * n
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(define (square n) (* n n))
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$\text{square}(2) \equiv (\lambda x.(x * x))(2) \rightsquigarrow 2 * 2$

In type theory as a foundation for mathematics:

- All the rules are just “axioms” that give meaning to undefined words like “type” and “term”, out of which we can then build mathematics.
- One usually thinks of “types” as kind of like sets.
- We will consider them as more like “spaces”.

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As a calculus for a cartesian closed category:

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 - Any $Z \times A \rightarrow B$ has an exponential transpose $Z \rightarrow B^A$.

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 - Any $Z \times A \rightarrow B$ has an exponential transpose $Z \rightarrow B^A$.
- ③ If $a: A$ and $f: B^A$, then $f(a): B$.
 - The evaluation map $B^A \times A \rightarrow B$.

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Exactly **the (weak) universal property of an exponential object.**

Type theorists write these rules as follows.

$$\frac{(x: A) \vdash (b: B)}{\vdash (\lambda x. b: B^A)}$$

$$\frac{\vdash (f: B^A) \quad \vdash (a: A)}{\vdash (f(a): B)}$$

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Type theorists also write $A \rightarrow B$ instead of B^A , but this can be confusing when also talking about arrows in a category.

Basic principle

There is a natural correspondence between

- ① **Programming:** ways to build datatypes in a computer
- ② **Foundations:** coherent sets of inference rules for type theory
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Therefore, if we can formalize a piece of mathematics inside of type theory, then

- it can be understood and verified by a computer, and
- it can be internalized in many other categories.

Informal mathematics

- We have the notion of a **group**: a set G with an element $e \in G$ and a binary operation satisfying certain axioms.
- We can prove theorems about groups, such as that inverses are unique: if $xy = e = xy'$, then $y = y'$.

We can also **formalize** this in ZFC, or in type theory, or in any other precise foundational system.

Example: Group objects

Internal mathematics

- A **group object** in a category is an object G with $e: 1 \rightarrow G$ and $m: G \times G \rightarrow G$, such that some diagrams commute:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times 1} & G \times G \\ 1 \times m \downarrow & & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array} \quad \text{etc.}$$

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- In sets: a group.
- In topological spaces: a topological group.
- In manifolds: a Lie group.
- In schemes: an algebraic group.
- In rings^{op}: a Hopf algebra.
- In sheaves: a sheaf of groups.

Example: Internalizing groups

Taking the informal notion of a group and formalizing it in type theory, we have a type G and terms

$$\vdash (e : G) \quad (x : G), (y : G) \vdash (x \cdot y : G)$$

satisfying appropriate axioms.

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The rules for interpreting type theory in categories give us:

- There is an **automatic** and **general** method which “extracts” or “compiles” the above formalization into **the notion of a group object in a category**.

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- There is an automatic and general method which “extracts” or “compiles” the above formalization into the notion of a group object in a category.
- Any theorem about ordinary groups that we can formalize in type theory likewise “compiles” to a **theorem about group objects in any category**.

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We can use “set-theoretic” reasoning with “elements” to prove “arrow-theoretic” facts about arbitrary categories.

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Coproduct types

Recall: every type constructor comes with rules for

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then $\text{case}(p, c_A, c_B): C$.

```
switch(p) {  
  if p is inl(x):  
    do cA with x  
  if p is inr(y):  
    do cB with y  
}
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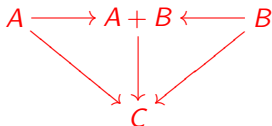
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- ② If $a: A$, then $\text{inl}(a): A + B$. If $b: B$, then $\text{inr}(b): A + B$.
 - Morphisms $\text{inl}: A \rightarrow A + B$ and $\text{inr}: B \rightarrow A + B$.

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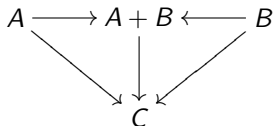
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Exactly **the (weak) universal property of a coproduct.**

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Exercise #2

Exercise

Define the empty type \emptyset .

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Aside: Polarity

- A **negative type** is characterized by **eliminations**.
 - We **eliminate** a term in some specified way.
 - We **introduce** a term by saying what it does when eliminated.
 - **Computation** follows the instructions of the introduction.
 - *Examples: function types B^A , products $A \times B$*
- A **positive type** is characterized by **introductions**.
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type theory	\longleftrightarrow	category theory
positive types	\longleftrightarrow	“from the left” universal properties
negative types	\longleftrightarrow	“from the right” universal properties

All universal properties expressible in type theory must be **stable** under products/pullbacks (i.e. adding unused variables).

Details that I am not mentioning (yet)

- Uniqueness in universal properties
- η -conversion rules
- Function extensionality
- Dependent eliminators
- Some types have both positive and negative versions
- Universe types (unpolarized)
- Eager and lazy evaluation
- Structural rules
- Coherence issues

Some of these will come up later.

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Type theory versus set theory

Set theory

Logic

$\wedge, \vee, \Rightarrow, \neg, \forall, \exists$

Sets

$\times, +, \rightarrow, \Pi, \Sigma$

$x \in A$ is a proposition

Type theory

Types

$\times, +, \rightarrow, \Pi, \Sigma$

Logic

$\wedge, \vee, \Rightarrow, \neg, \forall, \exists$

$x : A$ is a typing judgment

Propositions as some types

Basic principle

We identify a proposition P with the **subsingleton**

$$\{ \star \mid P \text{ is true} \}$$

(That is, $\{\star\}$ if P is true, \emptyset if P is false.)

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- To prove P is equivalent to exhibit an element of it.
- Proofs are just a particular sort of typing judgment:

$$(x_1 : P_1), \dots, (x_n : P_n) \vdash (q : Q)$$

“Under hypotheses P_1, P_2, \dots, P_n ,
the conclusion Q is provable.”

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q is a **proof term**, which records how each hypothesis was used.

The Curry-Howard correspondence

Restricted to subsingletons, the rules of type theory tell us **how to construct valid proofs**. This includes:

- 1 How to construct new propositions.
- 2 How to prove such propositions.
- 3 How to use such propositions to prove other propositions.
- 4 (Computation rules are less meaningful for subsingletons.)

Types	\longleftrightarrow	Propositions
$A \times B$	\longleftrightarrow	P and Q
$A + B$	\longleftrightarrow	P or Q
B^A	\longleftrightarrow	P implies Q
unit	\longleftrightarrow	\top (true)
\emptyset	\longleftrightarrow	\perp (false)

Function types, acting on subsingletons, become **implication**.

- ① If P and Q are propositions, then so is $P \Rightarrow Q$.
- ② If assuming P , we can prove Q , then we can prove $P \Rightarrow Q$.
- ③ If we can prove P and $P \Rightarrow Q$, then we can prove Q .

Cartesian products, acting on subsingletons, become **conjunction**.

- ① If P and Q are propositions, so is " P and Q ".
- ② If P is true and Q is true, then so is " P and Q ".
- ③ If " P and Q " is true, then P is true.
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The proof term

$$(f : P \Rightarrow (Q \text{ and } R)) \vdash (\lambda x. \text{fst}(f(x)) : P \Rightarrow Q)$$

encodes the following informal proof:

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If P implies Q and R , then P implies Q .

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This is how **type-checking a program** can **verify a proof**.

Subterminal objects

What does logic look like in a category?

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- In **Set** ^{D} : cosieves in D .
- In **Sh**(X): open subsets of X .

Problem

Not all operations preserve subsingletons.

- $A \times B$ is a subsingleton if A and B are
- B^A is a subsingleton if A and B are

But:

- $A + B$ is not generally a subsingleton, even if A and B are.

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Solution

The **support** of A is a “reflection” of A into subsingletons.

Thus “ P or Q ” means the support of $P + Q$.

I’ll explain the type constructor that does this on Friday.

Intuitionistic logic

We define the **negation** of P by

$$\neg P := (P \Rightarrow \perp).$$

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By itself, it is weaker than classical logic. But...

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- 3 A weaker logic means a wider validity (in more categories).

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Examples

- **Set** ^{D} has classical logic $\iff D$ is a groupoid.
- **Sh**(X) has classical logic \iff every open set in X is closed.

Exercise #3

Exercise

Write a program that proves $\neg\neg(A \text{ or } \neg A)$.

Other ways to interpret logic in type theory:

- Don't require “proposition types” to be subsingletons.
- Keep propositions as a separate “sort” from types.

- 1 Type theory and category theory
- 2 Type constructors and universal properties
- 3 Type theory and logic
- 4 Predicate logic and dependent types**
- 5 Equality types

For logic we need more than **connectives**

“and”, “or”, “implies”, “not”

we need **quantifiers**:

“for all $x \in X$ ”, “there exists an $x \in X$ such that”

Predicate logic

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we need quantifiers:

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First question

Before forming “there exists an $x \in X$ such that $P(x)$ ”, we need a notion of **predicate**: a “function” P from X to propositions.

Predicates and dependent types

If propositions are subsingleton types, then predicates must be **dependent types**: types that vary over some other type.

A dependent type judgment

$$(x : A) \vdash (B(x) : \text{Type})$$

means that for any particular $x : A$, we have a type $B(x)$. If each $B(x)$ is a subsingleton, then this is a predicate.

Examples of dependent types

$(y: \text{Year}), (m: \text{Month}) \vdash (\text{Day}(y, m) : \text{Type})$

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$(x: A), (y: A) \vdash ((x = y) : \text{Type})$

$(n: \mathbb{N}), (x: \mathbb{N}), (y: \mathbb{N}), (z: \mathbb{N}) \vdash ((x^n + y^n = z^n) : \text{Type})$

Universe types

The syntax

$$(x : A) \vdash (B(x) : \text{Type})$$

looks like there is a type called “Type” that $B(x)$ is an element of!

This is called a **universe type**: its *elements* are *types*.

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- “Type : Type” leads to paradoxes, but we can have a hierarchy

$$\text{Type}_0 : \text{Type}_1 : \text{Type}_2 : \dots$$

Dependent types in categories

In category theory, a dependent type “ $(x : A) \vdash (B(x) : \text{Type})$ ” is:

- ① A map $B \rightarrow A$, where $B(x)$ is the **fiber** over $x : A$; OR
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$$\begin{array}{ccc} B & \longrightarrow & \widetilde{\text{Type}} \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & \text{Type} \end{array}$$

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- B is a **predicate** if $B \rightarrow A$ is **monic**.

Dependent products

A proof of “ $\forall x: A, P(x)$ ” assigns, to each $a: A$, a proof of $P(a)$.
In general, we have the **dependent product**:

- 1 If $(x: A) \vdash (B(x) : \text{Type})$, there is a type $\prod_{x: A} B(x)$.
- 2 If $(x: A) \vdash (b : B(x))$, then $\lambda x. b : \prod_{x: A} B(x)$.
- 3 If $a: A$ and $f : \prod_{x: A} B(x)$, then $f(a) : B(a)$.
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Remark

If $B(x)$ is independent of x , then $\prod_{x: A} B(x)$ is just B^A .

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Dependent sums and products in categories

- Pullback of a dependent type “ $(y: B) \vdash (P(y) : \text{Type})$ ” along $f: A \rightarrow B$:

$$\begin{array}{ccc} f^*B & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

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$$\exists_f \dashv f^* \dashv \forall_f$$

(an insight due originally to Lawvere)

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Thinking about fibers leads us to conclude that

Eq_A should be represented by the **diagonal** $A \rightarrow A \times A$.

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(On Friday: a general framework which produces these rules.)

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Homotopical equality

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