Type theory and category theory

Michael Shulman

http://www.math.ucsd.edu/~mshulman/hottminicourse2012/

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The three faces of type theory

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2. A foundation for mathematics.
3. A calculus for category theory.

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2. A foundation for mathematics.
3. A calculus for category theory.

1 + 2 : A computable foundation for mathematics.
2 + 3 : A way to internalize mathematics in categories.
1 + 3 : A categorical description of programming semantics.

$$
\begin{tikzcd}
& \text{programming} & \\
\text{categories} \arrow[ur, leftrightarrow] \arrow[rr, leftrightarrow] & & \text{logic} \arrow[ul, leftrightarrow]
\end{tikzcd}
$$

The three faces of homotopy type theory

1. A programming language.
2. A foundation for mathematics based on homotopy theory.
3. A calculus for $(\infty, 1)$-category theory.

1 + 2 : A computable foundation for homotopical mathematics.
2 + 3 : A way to internalize homotopical mathematics in categories.
1 + 3 : A categorical description of programming semantics.

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 & \text{programming} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
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Minicourse plan

- Today: Type theory, logic, and category theory
- Wednesday: Homotopy theory in type theory
- Thursday: Type theory in $(\infty, 1)$-categories
- Friday: Current frontiers

Outline

Typing judgments

Type theory consists of rules for manipulating typing judgments:

$$(x_1 : A_1), (x_2 : A_2), \ldots, (x_n : A_n) \vdash (b : B)$$

- The x_i are variables, while b stands for an arbitrary expression.
- The turnstile $\vdash$ and commas are the "outermost" structure.

This should be read as:

In the context of variables x_1 of type A_1, x_2 of type A_2, $\ldots$, and x_n of type A_n, the expression b has type B.

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The meanings of a typing judgment

$$(x_1 : A_1), (x_2 : A_2), \ldots (x_n : A_n) \vdash (b : B)$$

1. Programming: A_i, B are datatypes (`int`, `float`, …); b is an expression of type B involving variables x_i of types A_i.
2. Foundations: A_i, B are "sets", b specifies a way to construct an element of B given elements x_i of A_i.
3. Category theory: A_i, B are objects, b specifies a way to construct a morphism $\prod_i A_i \to B$.

Type constructors

The rules of type theory come in packages called type constructors. Each package consists of:

1. Formation: a way to construct new types.
2. Introduction: ways to construct terms of these types.
3. Elimination: ways to use them to construct other terms.
4. Computation: what happens when we follow 2 by 3.

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Type theory as programming

$$\mathsf{square} := \lambda x.(x * x)$$

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int square(int x) { return (x * x); }
```

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def square(x):
  return (x * x)
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square :: Int -> Int
square x = x * x
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fun square (n:int):int = n * n
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(define (square n) (* n n))
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$$\text{square}(2) \;\equiv\; (\lambda x.(x * x))(2) \;\rightsquigarrow\; 2 * 2$$

Type constructors: as foundations

In type theory as a foundation for mathematics:

- All the rules are just "axioms" that give meaning to undefined words like "type" and "term", out of which we can then build mathematics.
- One usually thinks of "types" as kind of like sets.
- We will consider them as more like "spaces".

Type constructors: in categories

As a calculus for a cartesian closed category:

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   - Any $Z \times A \to B$ has an exponential transpose $Z \to B^A$.

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Exactly the (weak) universal property of an exponential object.

Inference rules

Type theorists write these rules as follows.

$$\frac{(x\colon A) \vdash (b\colon B)}{\vdash (\lambda x.b\colon B^A)}$$

$$\frac{\vdash (f\colon B^A) \qquad \vdash (a\colon A)}{\vdash (f(a)\colon B)}$$

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Type theorists also write $A \to B$ instead of B^A, but this can be confusing when also talking about arrows in a category.

Internal logic

Basic principle

There is a natural correspondence between

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Therefore, if we can formalize a piece of mathematics inside of type theory, then

- it can be understood and verified by a computer, and
- it can be internalized in many other categories.

Example: Groups

Informal mathematics

- We have the notion of a group: a set G with an element $e \in G$ and a binary operation satisfying certain axioms.
- We can prove theorems about groups, such as that inverses are unique: if $xy = e = xy'$, then $y = y'$.

We can also formalize this in ZFC, or in type theory, or in any other precise foundational system.

Example: Group objects

Internal mathematics

- A group object is a category is an object G with $e\colon 1 \to G$ and $m\colon G \times G \to G$, such that some diagrams commute:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
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\qquad \text{etc.}
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- In sets: a group.
- In topological spaces: a topological group.
- In manifolds: a Lie group.
- In schemes: an algebraic group.
- In rings^{op}: a Hopf algebra.
- In sheaves: a sheaf of groups.

Example: Internalizing groups

Taking the informal notion of a group and formalizing it in type theory, we have a type G and terms

$$\vdash (e \colon G) \qquad\qquad (x \colon G), (y \colon G) \vdash (x \cdot y : G)$$

satisfying appropriate axioms.

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- There is an automatic and general method which "extracts" or "compiles" the above formalization into the notion of a group object in a category.

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- There is an automatic and general method which "extracts" or "compiles" the above formalization into the notion of a group object in a category.
- Any theorem about ordinary groups that we can formalize in type theory likewise "compiles" to a theorem about group objects in any category.

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We can use "set-theoretic" reasoning with "elements" to prove "arrow-theoretic" facts about arbitrary categories.

Outline

Coproduct types

Recall: every type constructor comes with rules for

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1. If A and B are types, then there is a new type $A + B$.
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Coproduct types: as programming

3. If $p\colon A+B$ and $(x\colon A) \vdash (c_A\colon C)$ and $(y\colon B) \vdash (c_B\colon C)$, then $\mathsf{case}(p, c_A, c_B)\colon C$.

```
switch(p) {
  if p is inl(x):
    do cA with x
  if p is inr(y):
    do cB with y
}
```

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4. $\mathsf{fst}(a, b)$ computes to a, and $\mathsf{snd}(a, b)$ computes to b.

Exercise #2

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Aside: Polarity

- A negative type is characterized by eliminations.
  - We eliminate a term in some specified way.
  - We introduce a term by saying what it does when eliminated.
  - Computation follows the instructions of the introduction.
  - *Examples: function types B^A, products $A \times B$*
- A positive type is characterized by introductions.
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type theory	$\longleftrightarrow$	category theory
positive types	$\longleftrightarrow$	"from the left" universal properties
negative types	$\longleftrightarrow$	"from the right" universal properties

All universal properties expressible in type theory must be stable under products/pullbacks (i.e. adding unused variables).

Details that I am not mentioning (yet)

- Uniqueness in universal properties
- η-conversion rules
- Function extensionality
- Dependent eliminators
- Some types have both positive and negative versions
- Universe types (unpolarized)
- Eager and lazy evaluation
- Structural rules
- Coherence issues

Some of these will come up later.

Outline

Type theory versus set theory

Set theory

Logic

$\wedge, \vee, \Rightarrow, \neg, \forall, \exists$

Sets

$\times, +, \rightarrow, \prod, \sum$

$x \in A$ is a proposition

Type theory

Types

$\times, +, \rightarrow, \prod, \sum$

Logic

$\wedge, \vee, \Rightarrow, \neg, \forall, \exists$

$x : A$ is a typing judgment

Propositions as some types

Basic principle

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- Proofs are just a particular sort of typing judgment:

$$(x_1 : P_1), \ldots, (x_n : P_n) \vdash (q : Q)$$

"Under hypotheses P_1, P_2, $\ldots$, P_n,
the conclusion Q is provable."

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q is a proof term, which records how each hypothesis was used.

The Curry-Howard correspondence

Restricted to subsingletons, the rules of type theory tell us how to construct valid proofs. This includes:

1. How to construct new propositions.
2. How to prove such propositions.
3. How to use such propositions to prove other propositions.
4. (Computation rules are less meaningful for subsingletons.)

Types	$\longleftrightarrow$	Propositions
$A \times B$	$\longleftrightarrow$	P and Q
$A + B$	$\longleftrightarrow$	P or Q
B^A	$\longleftrightarrow$	P implies Q
unit	$\longleftrightarrow$	$\top$ (true)
$\emptyset$	$\longleftrightarrow$	$\bot$ (false)

Implication

Function types, acting on subsingletons, become implication.

1. If P and Q are propositions, then so is $P \Rightarrow Q$.
2. If assuming P, we can prove Q, then we can prove $P \Rightarrow Q$.
3. If we can prove P and $P \Rightarrow Q$, then we can prove Q.

Conjunction

Cartesian products, acting on subsingletons, become conjunction.

1. If P and Q are propositions, so is "P and Q".
2. If P is true and Q is true, then so is "P and Q".
3. If "P and Q" is true, then P is true.
   If "P and Q" is true, then Q is true.

Proof terms

The proof term

$$\big(f : P \Rightarrow (Q \text{ and } R)\big) \ \vdash \ \big(\lambda x.\mathsf{fst}(f(x)) : P \Rightarrow Q\big)$$

encodes the following informal proof:

Theorem

If P implies Q and R, then P implies Q.

Proof.

- Suppose P.
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- Therefore, P implies Q. $\square$

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This is how type-checking a program can verify a proof.

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What does logic look like in a category?

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- In $\mathbf{Set}^{D}$: cosieves in D.
- In $\mathbf{Sh}(X)$: open subsets of X.

Supports

Problem

Not all operations preserve subsingletons.

- $A \times B$ is a subsingleton if A and B are
- B^A is a subsingleton if A and B are

But:

- $A + B$ is not generally a subsingleton, even if A and B are.

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But:

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Solution

The support of A is a "reflection" of A into subsingletons.

Thus "P or Q" means the support of $P + Q$.
I'll explain the type constructor that does this on Friday.

Intuitionistic logic

We define the negation of P by

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What we have is called intuitionistic or constructive logic.
By itself, it is weaker than classical logic. But...

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Examples

- $\mathbf{Set}^D$ has classical logic $\iff$ D is a groupoid.
- $\mathbf{Sh}(X)$ has classical logic $\iff$ every open set in X is closed.

Exercise #3

Exercise

Write a program that proves $\neg\neg(A \text{ or } \neg A)$.

Details that I am not mentioning

Other ways to interpret logic in type theory:

- Don't require "proposition types" to be subsingletons.
- Keep propositions as a separate "sort" from types.

Outline

Predicate logic

For logic we need more than connectives

"and", "or", "implies", "not"

we need quantifiers:

"for all $x \in X$", "there exists an $x \in X$ such that"

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$$\text{“for all } x \in X\text{”},\ \text{“there exists an } x \in X \text{ such that”}$$

First question

Before forming "there exists an $x \in X$ such that $P(x)$", we need a notion of predicate: a "function" P from X to propositions.

Predicates and dependent types

If propositions are subsingleton types, then predicates must be dependent types: types that vary over some other type.

A dependent type judgment

$$(x : A) \vdash (B(x) : \mathrm{Type})$$

means that for any particular $x : A$, we have a type $B(x)$. If each $B(x)$ is a subsingleton, then this is a predicate.

Examples of dependent types

$$(y\colon \mathrm{Year}), (m\colon \mathrm{Month}) \;\vdash\; (\mathrm{Day}(y, m) : \mathrm{Type})$$

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$$(x\colon A), (y\colon A) \vdash ((x = y) : \mathrm{Type})$$

$$(n\colon \mathbb{N}), (x\colon \mathbb{N}), (y\colon \mathbb{N}), (z\colon \mathbb{N}) \vdash ((x^n + y^n = z^n) : \mathrm{Type})$$

Universe types

The syntax

$$(x : A) \vdash (B(x) : \mathsf{Type})$$

looks like there is a type called "Type" that $B(x)$ is an element of!

This is called a universe type: its *elements* are *types*.

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- "Type: Type" leads to paradoxes, but we can have a hierarchy

$$\mathrm{Type}_0 : \mathrm{Type}_1 : \mathrm{Type}_2 : \cdots$$

Dependent types in categories

In category theory, a dependent type "$(x\colon A) \vdash (B(x) : \mathrm{Type})$" is:

1. A map $B \to A$, where $B(x)$ is the fiber over $x\colon A$; OR
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- B is a predicate if $B \to A$ is monic.

Dependent products

A proof of "$\forall x\colon A, P(x)$" assigns, to each $a\colon A$, a proof of $P(a)$.
In general, we have the dependent product:

1. If $(x\colon A) \;\vdash\; (B(x) : \text{Type})$, there is a type $\prod_{x\colon A} B(x)$.
2. If $(x\colon A) \;\vdash\; (b : B(x))$, then $\lambda x.b\colon \prod_{x\colon A} B(x)$.
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Remark

If $B(x)$ is independent of x, then $\prod_{x\colon A} B(x)$ is just B^A.

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If $B(x)$ is independent of x, then $\sum_{x\colon A} B(x)$ reduces to $A \times B$.

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$\prod_{x:A} B(x)$	$\longleftrightarrow$	$\forall x : A, P(x)$
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- $\prod_{x:A} B(x)$ is a subsingleton if each $B(x)$ is.
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- $\sum_{x:A} B(x)$ is not, so we use its support, as with "or".
- If B is a subsingleton, $\sum_{x:A} B(x)$ is "$\{x: A \mid B(x)\}$".

Dependent sums and products in categories

- Pullback of a dependent type "$(y \colon B) \vdash (P(y) : \mathrm{Type})$" along $f \colon A \to B$:

$$\begin{tikzcd} f^*B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

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$$\exists_f \dashv f^* \dashv \forall_f$$

(an insight due originally to Lawvere)

Outline

Equality in categories

To formalize mathematics, we need to talk about equality.

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Thinking about fibers leads us to conclude that

Eq_A should be represented by the diagonal $A \to A \times A$.

Equality in type theory

Equality is just another (positive) type constructor.

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4. $J(d; a, a, \mathsf{refl}_a)$ computes to $d(a)$.

(On Friday: a general framework which produces these rules.)

Homotopical equality

Two Big Important Facts

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$$\begin{tikzcd}
\boxed{\text{Homotopy theory}} \arrow[r, shift left, "\text{homotopy (type theory)}"] & \boxed{\text{Type theory}} \arrow[l, shift left, "\text{(homotopy type) theory}"]
\end{tikzcd}$$