# Homotopy theory in type theory

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## Review of type theory

- Type theory consists of rules for deriving typing judgments:

$$(x_1 : A_1), (x_2 : A_2), \ldots, (x_n : A_n) \vdash (b : B)$$

- The rules come in "packages" called type constructors.
- Each type constructor has four groups of rules: formation, introduction, elimination, and computation.
- Categorically: types are objects, terms are morphisms.
- Each type constructor corresponds to a categorical universal property.

## Dependent eliminators

When we introduce predicates and dependent types, the eliminators of other types need to be generalized.

**Example**

- Suppose $(z : A + B) \vdash (P(z) : \mathrm{Type})$ is a predicate on $A + B$.
- We should be able to prove $P$ by cases.
  1. Prove $(x : A) \vdash (p_A : P(\mathrm{inl}(x)))$.
  2. Prove $(y : B) \vdash (p_B : P(\mathrm{inr}(y)))$.
  3. Conclude $(z : A + B) \vdash (\mathrm{case}(z; p_A, p_B) : P(z))$.
- This looks like the "case split" eliminator for $A + B$, but the output type $P(z)$ depends on the element $z$ that we are case-analyzing.

Therefore: we strengthen the elimination rules.

## Dependent eliminators

**Before**

Suppose $A$, $B$, and $C$ are types.

If $(x : A) \vdash (c_A : C)$ and $(y : B) \vdash (c_B : C)$, then for $p : A + B$ we have $\mathrm{case}(p, c_A, c_B) : C$.

**After**

Suppose $A$ and $B$ are types, and

$$(z : A + B) \vdash (C(z) : \mathrm{Type})$$

is a dependent type.

If $(x : A) \vdash (c_A : C(\mathrm{inl}(x)))$ and $(y : B) \vdash (c_B : C(\mathrm{inr}(y)))$, then for $p : A + B$ we have $\mathrm{case}(p, c_A, c_B) : C(p)$.

## Dependent eliminators in categories

$$
\begin{tikzcd}
 & & C \arrow[d] \\
A \arrow[urr] \arrow[rr] & & A+B \arrow[u, dashed, bend left] \\
 & B \arrow[uur] \arrow[ur] &
\end{tikzcd}
$$

## Dependent eliminators imply uniqueness

**Theorem**

*Suppose $f, g\colon C^{A+B}$ and that*

- *for all $a\colon A$, we have $f(\mathrm{inl}(a)) = g(\mathrm{inl}(a))$, and*
- *for all $b\colon B$, we have $f(\mathrm{inr}(b)) = g(\mathrm{inr}(b))$.*

*Then for all $z\colon A+B$, we have $f(z) = g(z)$.*

**Proof.**

Consider the dependent type

$$(z\colon A+B) \vdash (f(z) = g(z) : \mathrm{Type})$$

By the dependent eliminator for $A+B$, to construct a term of this type, it suffices to construct terms

$$(a\colon A) \vdash (e_A : f(\mathrm{inl}(a)) = g(\mathrm{inl}(a)))$$
$$(b\colon B) \vdash (e_B : f(\mathrm{inr}(b)) = g(\mathrm{inr}(b)))$$

□

## Function extensionality

It's more difficult to give a dependent eliminator for function types. Instead, we assert function extensionality directly as an axiom.

$$\left(f, g\colon B^A\right) \vdash \left(\mathrm{funext} : \left(\textstyle\prod_{x\colon A}(f(x) = g(x))\right) \to (f = g)\right)$$

**Remarks**

- Today I'll use both $B^A$ and $A \to B$ for the function type.
- Later: more homotopical versions of both kinds of uniqueness.

## Equality types

Equality types (or identity types) are a "positive type" (determined by the introduction rule):

1. For any type $A$ and $a\colon A$ and $b\colon A$, there is a type $(a = b)$.
2. For any $a\colon A$, we have $\mathrm{refl}_a : (a = a)$.
3. Suppose $C(x, y, p)$ is a type dependent on three variables $x, y\colon A$ and $p\colon (x = y)$. Suppose moreover that for any $x\colon A$ we have an element $d(x) : C(x, x, \mathrm{refl}_x)$. Then for any $x, y, p$ we have an element $J(d; x, y, p) : C(x, y, p)$.
4. $J(d; a, a, \mathrm{refl}_a)$ computes to $d(a)$.

Informally, 3 says

**Elimination on equality**

In order to do something with an arbitrary $p\colon (x = y)$,
it suffices to consider the case of $\mathrm{refl}_x : (x = x)$.

## Equality is symmetric

**Theorem**

*Suppose $p\colon (x = y)$. Then $p^{-1} : (y = x)$.*

**Proof.**

By elimination, we may assume that $p$ is $\mathsf{refl}_x : (x = x)$. But in this case, we can take $p^{-1}$ to also be $\mathsf{refl}_x : (x = x)$. □

Just as in the cases of the dependent eliminator for coproducts, the desired conclusion $C(z)$ becomes $C(\mathsf{inl}(a))$ and $C(\mathsf{inr}(b))$, when we eliminate $p$ the desired conclusion $(y = x)$ becomes $(x = x)$.

## Equality is transitive

**Theorem**

*Suppose $p\colon (x = y)$ and $q\colon (y = z)$. Then $p * q : (x = z)$.*

**Proof.**

By elimination, we may assume that $p$ is $\mathsf{refl}_x : (x = x)$. But in this case, we have $q\colon (x = z)$, so we can take $p * q$ to be just $q$. □

We could equally well have eliminated $q$, or both $p$ and $q$.

## Paths

We treat types as spaces/$\infty$-groupoids/homotopy types, and we think of terms $p\colon (x = y)$ as paths $x \rightsquigarrow y$.

- Reflexivity becomes the constant path $\mathsf{refl}_x\colon x \rightsquigarrow x$.
- Transitivity becomes concatenation $x \overset{p*q}{\rightsquigarrow} z$ of $x \overset{p}{\rightsquigarrow} y \overset{q}{\rightsquigarrow} z$.
- Symmetry becomes reversal $y \overset{p^{-1}}{\rightsquigarrow} x$ of $x \overset{p}{\rightsquigarrow} y$.

But now there is more to say.

- Concatenation is associative: $\alpha_{p,q,r} : ((p * q) * r = p * (q * r))$.

## 2-paths

The “associator” $\alpha_{p,q,r}$ is coherent:

$$\begin{tikzcd}
 & (p*q)*(r*s) \arrow[dr, rightsquigarrow] & \\
((p*q)*r)*s \arrow[ur, rightsquigarrow] \arrow[d, rightsquigarrow] & & p*(q*(r*s)) \\
(p*(q*r))*s \arrow[dr, rightsquigarrow] & & \\
 & p*((q*r)*s) \arrow[uur, rightsquigarrow] &
\end{tikzcd}$$

. . . or more precisely, there is a path between those two concatenations. . .

. . . which then has to be coherent. . .

## $\infty$-groupoids

**Theorem (Lusmdaine,Garner–van den Berg)**

*The terms belonging to the iterated identity types of any type $A$ form an $\infty$-groupoid.*

$$\begin{tikzcd} \boxed{\text{Homotopy theory}} \arrow[r, "\text{homotopy (type theory)}"] & \boxed{\text{Type theory}} \end{tikzcd}$$

Note: Uses Batanin-Leinster $\infty$-groupoids (can also be done with simplicial versions).

## Mapping on paths

Given $f : A \to B$, $x, y : A$, and a path $p : (x = y)$, we have an image path

$$\mathsf{map}(f, p) : (f(x) = f(y))$$

defined by eliminating on $p$:

- If $p$ is $\mathsf{refl}_x$, then $\mathsf{map}(f, p) :\equiv \mathsf{refl}_{f(x)}$.

## Transporting along paths

Given $x, y : A$, $p : (x = y)$, and $B$ dependent on $A$, we have the operation of transporting along $p$

$$\mathsf{trans}(p, -) : B(x) \to B(y).$$

defined by eliminating on $p$:

- If $p$ is $\mathsf{refl}_x$, then $\mathsf{trans}(p, -)$ is the identity map of $B(x)$.

**Interpretation**

We should view the map $B \to A$ as a fibration.
(In an $(\infty, 1)$-category, we can treat any map as a fibration.)

## Paths for type constructors

For any type built using a type constructor, we can characterize its paths in terms of paths in its input types.

**Example (Cartesian products)**

- From $p : (a_1 = a_2)$ and $q : (b_1 = b_2)$, we can build

$$(p, q) : ((a_1, b_1) = (a_2, b_2))$$

- Given $z_1, z_2 : A \times B$ and $r : (z_1 = z_2)$, we have

$$\mathsf{map}(\mathsf{fst}, r) : (\mathsf{fst}(z_1) = \mathsf{fst}(z_2))$$

$$\mathsf{map}(\mathsf{snd}, r) : (\mathsf{snd}(z_1) = \mathsf{snd}(z_2))$$

## Paths in dependent sums

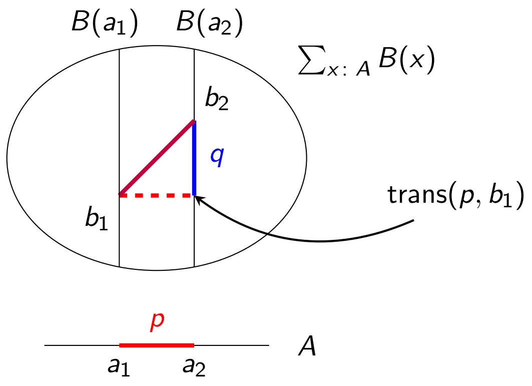
Suppose $a_1, a_2 \colon A$ and $b_1 \colon B(a_1)$ and $b_2 \colon B(a_2)$. A path

$$(a_1, b_1) = (a_2, b_2)$$

in $\sum_{x\colon A} B(x)$ should consist of

- A path $p \colon (a_1 = a_2)$ in $A$, and...
- what?
  - The expression $(b_1 = b_2)$ is ill-formed, since $b_1$ and $b_2$ have different types.
  - Instead we can use $q \colon (\mathsf{trans}(p, b_1) = b_2)$.

## Paths in dependent sums




- In a fibration, we can lift the path $p$ starting at $b_1$.
- We choose one lift and call its endpoint $\mathsf{trans}(p, b_1)$.
- Any other lift of $p$ is determined by a path in the fiber $B(a_2)$.

## Subsingletons in homotopy theory

Recall that logic is type theory restricted to subsingletons.

In homotopy type theory, we interpret "subsingleton" homotopically:

**Theorem**

*For an object $P$ in an $(\infty, 1)$-category with products, TFAE:*

1. *Each space* $\mathrm{Hom}(X, P)$ *is empty or contractible.*
2. *Any two morphisms $X \rightrightarrows P$ are homotopic.*
3. *The diagonal $P \to P \times P$ has a section.*
4. *The diagonal $P \to P \times P$ is an equivalence.*

## h-Propositions

**Definition**

A type $P$ is a proposition (or h-proposition or h-prop) if we have

$$(x \colon P), (y \colon P) \vdash (p : (x = y))$$

$$\begin{tikzcd} P \arrow[d] \\ P \times P \arrow[u, bend left, dashed] \end{tikzcd}$$

These are the "subsingletons" of homotopy type theory.

## Building h-props

What ways do we have to obtain h-props?

- Most type constructors preserve h-props.
- For others ($+$ and $\sum$), we intend to apply "support".
- $(x = y)$ is not generally an h-prop, but has a support:
  - $(x = y)$ is the type of paths from $x$ to $y$.
  - $\mathrm{supp}(x = y)$ is the assertion: there exists a path from $x$ to $y$.
- For some types $A$, all equalities $(x = y)$ are h-props.
  - These are called sets or h-sets.
  - Certain types are always sets (e.g. $\mathbb{N}$, on Friday).
- But can we say anything homotopy-theoretic with this logic?

## Internalizing h-props

How can we say in type theory "$A$ is an h-prop"?

$$\mathrm{isProp}(A) :\equiv \mathrm{supp}\left(\prod_{x:A}\prod_{y:A}(x = y)\right) \qquad ?$$

## Internalizing h-props

How can we say in type theory "$A$ is an h-prop"?

$$\mathrm{isProp}(A) :\equiv \qquad \prod_{x:A}\prod_{y:A}(x = y) \qquad !$$

This is already an h-prop!

**Theorem**

*For any $A$, we can construct a term in*

$$\mathrm{isProp}(\mathrm{isProp}(A)).$$

## Internalizing h-props

**Theorem**

*For any $A$,* $\mathrm{isProp}(\mathrm{isProp}(A))$.

**Proof.**

- Suppose $H, K : \mathrm{isProp}(A)$; we want $(H = K)$.
- By funext, suffices to show $H(x,y) = K(x,y)$ for all $x, y : A$.
- Now $\mathrm{map}(K(x,-), H(x,y))$ is a path in $\sum_z (x = z)$ from $K(x,x)$ to $K(x,y)$. In particular, it contains a path

$$\mathrm{trans}(H(x,y), K(x,x)) = K(x,y)$$

- Hence $H(x,y) * K(x,x) = K(x,y)$ (a fact).
- It suffices to prove $K(x,x) = \mathrm{refl}_x$.
  - The above argument with $H$ being $K$, and $y$ being $x$, yields $K(x,x) * K(x,x) = K(x,x)$.
  - Now cancel $K(x,x)$ (i.e. concatenate with $K(x,x)^{-1}$). $\square$

## Some subtleties

- We can loosely read $\prod_{x:A}\prod_{y:A}(x=y)$ as
  "for all $x, y: A$, we have a path $(x=y)$"
- But "for all $x, y: A$, there exists a path $(x=y)$" should be read to mean
  $$\prod_{x:A}\prod_{y:A}\mathsf{supp}(x=y)$$
  This asserts that "if $A$ is nonempty, then it is connected."
- In $\prod_{x:A}\prod_{y:A}(x=y)$, the assigned path $(x=y)$ must depend continuously on $x$ and $y$. This can be confusing until you get used to this meaning of "for all".

## Some subtleties

- Type theory is a formal system.
- We can and do (and must, in practice) use informal language to speak and think about it.
- This depends on certain conventions about the formal interpretation given to informal words, which are sometimes subtly different to those used for some other formal system (like set theory).
- Fortunately, we have a computer proof assistant to type-check our proofs and guarantee that we didn't screw up!

## Homotopy equivalences

**Definition**

A function $f: A \to B$ is a homotopy equivalence if there exists $g: B \to A$ and homotopies $g \circ f \sim \mathsf{id}_A$ and $f \circ g \sim \mathsf{id}_B$.

$$\mathsf{isHtpyEquiv}(f) :\equiv \mathsf{supp}\left(\sum_{g: B\to A}\Big((g\circ f = \mathsf{id}_A)\times(f\circ g=\mathsf{id}_B)\Big)\right)$$

This would not be an h-prop without supp. Can we avoid it?

## Back to bijections

A function $f: A \to B$ between sets is a bijection if

1. There exists $g: B \to A$ such that $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$.
2. OR: For each $b \in B$, the set $f^{-1}(b)$ is a singleton.
3. OR: There exists $g: B \to A$ such that $g \circ f = \mathsf{id}_A$ and also $h: B \to A$ such that $f \circ h = \mathsf{id}_B$.

## Voevodsky equivalences

### Definitions

The homotopy fiber of $f\colon A \to B$ at $b\colon B$ is

$$\mathrm{hfiber}(f,b) :\equiv \sum_{x\colon A} (f(x) = b).$$

A type $X$ is contractible if it is an inhabited h-prop:

$$\mathrm{isContr}(X) :\equiv \mathrm{isProp}(X) \times X$$

### Definition (Voevodsky)

$f$ is an equivalence if each $\mathrm{hfiber}(f,b)$ is contractible:

$$\mathrm{isEquiv}(f) :\equiv \prod_{b\colon B} \mathrm{isContr}(\mathrm{hfiber}(f,b))$$

This is an h-prop.

## H-isomorphisms

### Definition (Joyal)

$f\colon A \to B$ is an h-isomorphism if we have $g\colon B \to A$ and a homotopy $g \circ f \sim \mathrm{id}_A$, and also $h\colon B \to A$ and a homotopy $f \circ h \sim \mathrm{id}_B$.

$$\mathrm{isHIso}(f) :\equiv \left( \sum_{g\colon B \to A} (g \circ f = \mathrm{id}_A) \right) \times \left( \sum_{h\colon B \to A} (f \circ h = \mathrm{id}_B) \right)$$

This is also an h-prop.

## Adjoint equivalences

Given a homotopy equivalence, we can also ask for more coherence from $r\colon (g \circ f = \mathrm{id}_A)$ and $s\colon (f \circ g = \mathrm{id}_B)$.

(1a) For all $b\colon B$, we have $u(b)\colon (r(g(b)) = \mathrm{map}(g, s(b)))$.

(1b) For all $a\colon A$, we have $v(a)\colon (\mathrm{map}(f, r(a)) = s(f(a)))$.

(2a) For all $b\colon B$, we have $\ldots v(g(b)) \ldots \mathrm{map}(g, u(b)) \ldots$

(2b) For all $a\colon A$, we have $\ldots u(f(a)) \ldots \mathrm{map}(f, v(a)) \ldots$

$\vdots$

This gives an h-prop if we stop between any ($n$ a) and ($n$ b) (and then the rest can be constructed).

### Definition

$f$ is an adjoint equivalence if we have $g$, $r$, $s$, and $u$.

$$\mathrm{isAdjEquiv}(f) :\equiv \sum_{g\colon B \to A} \sum_{r\colon \ldots} \sum_{s\colon \ldots} \Big( r(g(b)) = \mathrm{map}(g, s(b)) \Big)$$

## All equivalences are the same

### Theorem

*The following are equivalent:*

1. *$f$ is a homotopy equivalence.*
2. *$f$ is a (Voevodsky) equivalence.*
3. *$f$ is a (Joyal) h-isomorphism.*
4. *$f$ is an adjoint equivalence.*

The last three are supp-free h-props, so we have equivalences

$$\mathrm{isEquiv}(f) \simeq \mathrm{isHIso}(f) \simeq \mathrm{isAdjEquiv}(f)$$

### Definition

The type of equivalences between $A, B\colon \mathrm{Type}$ is

$$\mathrm{Equiv}(A,B) :\equiv \sum_{f\colon A \to B} \mathrm{isEquiv}(f).$$

## The short five lemma

$$\begin{tikzcd} \mathrm{hfiber}(f) \arrow[r] \arrow[d, "r"] & A \arrow[r, "f"] \arrow[d, "s"] & B \arrow[d, "t"] \\ \mathrm{hfiber}(g) \arrow[r] & C \arrow[r, "g"'] & D \end{tikzcd}$$

**Theorem**

- *If $s$ and $t$ are equivalences, so is $r$.*
- *If $r$ and $t$ are equivalences, so is $s$.*

This is a theorem in type theory: $A$, $B$, $C$, $D$ are types and we have a proof term

$$\big(p_1 : \mathrm{isEquiv}(s)\big), \big(p_2 : \mathrm{isEquiv}(t)\big) \vdash \big(q : \mathrm{isEquiv}(r)\big)$$

## The $3 \times 3$ lemma

$$\begin{tikzcd} & \mathrm{hfiber}(h) \arrow[r, "r"] \arrow[d] & \mathrm{hfiber}(k) \arrow[d] \\ \mathrm{hfiber}(f) \arrow[r] \arrow[d, "s"'] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ \mathrm{hfiber}(g) \arrow[r] & C \arrow[r, "g"'] & D \end{tikzcd}$$

**Theorem**

*There is an equivalence* $\mathrm{hfiber}(r) \simeq \mathrm{hfiber}(s)$.

(Also a theorem in type theory.)

## Homotopical uniqueness

**Theorem**

*For any types $A$, $B$, $C$, the map*

$$\lambda f.\big(\lambda a.f(\mathrm{inl}(a)), \lambda b.f(\mathrm{inr}(b))\big) \ : \ C^{A+B} \to C^A \times C^B$$

*is an equivalence (using the dependent eliminator).*

The type $C^{A+B} \to C^A \times C^B$ should be more consistently (but less legibly) written:

$$(C^A \times C^B)^{C^{A+B}} \quad \text{or} \quad ((A+B) \to C) \to ((A \to C) \times (B \to C))$$

Awodey–Gambino–Sojakova have proven a much more general version of this, in the context we'll discuss on Friday.

## Homotopical function extensionality

For $f, g : B^A$, there is a term

$$\mathrm{happly} \ : \ \left((f = g) \to \prod_{a:A} (f(a) = g(a))\right)$$

defined by identity elimination:

$$\mathrm{happly}(\mathrm{refl}_f) := \lambda a.\mathrm{refl}_{f(a)}$$

**Theorem (Voevodsky)**

happly *is an equivalence (using the naive* funext*).*

Also works for dependent functions.

## Paths in the universe

The only type whose path-types we have not determined (up to equivalence, in terms of other path-spaces) is the universe "Type".

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathsf{Type}} \arrow[d] \\ A \arrow[r] & \mathsf{Type} \end{tikzcd}$$

If Type is the "classifying space" of types, then a path in Type should be an equivalence of types.

## The univalence axiom

For $A, B\colon \mathsf{Type}$, we have

$$\mathsf{pathToEquiv}_{A,B} \;:\; \big((A = B) \to \mathsf{Equiv}(A, B)\big)$$

defined by identity elimination.
Note: $(A = B)$ is a path-type of "Type".

**The Univalence Axiom (Voevodsky)**

For all $A, B$, the function $\mathsf{pathToEquiv}_{A,B}$ is an equivalence.

$$\prod_{A\colon \mathsf{Type}} \; \prod_{B\colon \mathsf{Type}} \mathsf{isEquiv}(\mathsf{pathToEquiv}_{A,B})$$

In particular, every equivalence yields a path between types.

## The meaning of univalence

**The meaning of univalence**

Given an equivalence $f\colon A \xrightarrow{\sim} B$, we can identify $A$ with $B$ along $f$.

In other words:

- When talking about $A$, $B$, and $f$, we "may as well assume" that $B$ is $A$, and $f$ is $1_A$.
- Or: equivalent types can be treated as identical.

**Proof.**

Use the inverse of pathToEquiv, then the eliminator of equality. □

This is something we do informally all the time in mathematics.
The univalence axiom gives it a precise formal expression.

## The uses of univalence

1. The homotopy theory is nontrivial (Type is not an h-set).
2. (Voevodsky) Univalence implies funext.
3. For any type $F$, the type

   $$\sum_{A\colon \mathsf{Type}} \mathsf{supp}(A = F)$$

   is the classifying space for bundles with fiber $F$.
4. Computing homotopy groups! (on Friday)
5. Many more . . .