Categorical models of homotopy type theory

Michael Shulman

12 April 2012

Homotopy type theory in higher categories

Recall:

homotopy type theory		$(\infty, 1)$-categories
$\times$, $+$ types	$\longleftrightarrow$	products, coproducts
equality types $(x = y)$	$\longleftrightarrow$	diagonals
$\prod$ types	$\longleftrightarrow$	local cartesian closure
univalent universe Type	$\longleftrightarrow$	object classifier

Two kinds of equality

Problem

Type theory is stricter than $(\infty, 1)$-categories.

In type theory, we have two kinds of "equality":

1. Equality witnessed by inhabitants of equality types ($=$ paths).
2. Computational equality: $(\lambda x.b)(a)$ evaluates to $b[a/x]$.

These play different roles: type checking depends on computational equality.

- if a evaluates to b, and $c : C(a)$, then also $c : C(b)$.
  - In particular, if a evaluates to b, then $\mathsf{refl}_b : (a = b)$.
- if $p : (a = b)$ and $c : C(a)$, then only $\mathsf{transport}(p, c) : C(b)$.

Two kinds of equality

But computational equality is also stricter.

Example

Composition is computationally strictly associative.

$$\begin{aligned} g \circ f &:= \lambda x. g(f(x)) \\ h \circ (g \circ f) &= \lambda x. h\Big((\lambda x. g(f(x)))(x)\Big) \rightsquigarrow \lambda x. h(g(f(x))) \\ (h \circ g) \circ f &= \lambda x. \Big(\lambda y. h(g(y))\Big)(f(x)) \rightsquigarrow \lambda x. h(g(f(x))) \end{aligned}$$

- This is the sort of issue that homotopy theorists are intimately familiar with!
- We need a model for $(\infty, 1)$-categories with (at least) a strictly associative composition law.

Display map categories

Forget everything you know about homotopy theory; let's see how the type theorists come at it.

Definition

A display map category is a category with

- A terminal object.
- A subclass of its morphisms called the display maps, denoted $P \twoheadrightarrow A$ or $P \rightarrowtriangle A$.
- Any pullback of a display map exists and is a display map.

- A display map $P \twoheadrightarrow A$ is a type dependent on A.
- A display map $A \twoheadrightarrow 1$ is a plain type (dependent on nothing).
- Pullback is substitution.

Dependent sums of display maps

$$(x\colon A) \vdash (B(x) : \mathrm{Type})$$

If the types $B(x)$ are the fibers of $B \twoheadrightarrow A$, their dependent sum $\sum_{x\colon A} B(x)$ should be the object B.

$$(x\colon A) \vdash (B(x) : \mathrm{Type}) \qquad \begin{tikzcd} B \arrow[d, two heads] \\ A \arrow[d, two heads] \\ 1 \end{tikzcd}$$

$$\vdash \left(\sum_{x\colon A} B(x) : \mathrm{Type} \right) \qquad \begin{tikzcd} B \arrow[d, two heads] \\ 1 \end{tikzcd}$$

Dependent sums in context

More generally:

$$(x\colon A), (y\colon B(x)) \vdash (C(x,y) : \mathrm{Type}) \qquad \begin{tikzcd} C \arrow[d, two heads] \\ B \arrow[d, two heads] \\ A \end{tikzcd}$$

$$(x\colon A) \vdash \left(\textstyle\sum_{y\colon B(x)} C(x,y) : \mathrm{Type}\right) \qquad \begin{tikzcd} C \arrow[d, two heads] \\ A \end{tikzcd}$$

Dependent sums $\longleftrightarrow$ display maps compose

Aside: adjoints to pullback

- In a category $\mathscr{C}$, if pullbacks along $f\colon A \to B$ exist, then the functor
$$f^*\colon \mathscr{C}/B \longrightarrow \mathscr{C}/A$$
has a left adjoint Σ_f given by composition with f.
- If f is a display map and display maps compose, then Σ_f restricts to a functor
$$(\mathscr{C}/A)_{\mathrm{disp}} \longrightarrow (\mathscr{C}/B)_{\mathrm{disp}}$$
implementing dependent sums.
- A right adjoint to f^*, if one exists, is an "object of sections". $\mathscr{C}$ is locally cartesian closed iff all such right adjoints Π_f exist.

Dependent products of display maps

$$(x\colon A), (y\colon B(x)) \vdash (C(x,y) : \mathrm{Type})$$

$$\begin{tikzcd} C \arrow[d, two heads] & \\ B \arrow[r, two heads] & A \end{tikzcd}$$

$$(x\colon A) \vdash \left(\prod_{y\colon B(x)} C(x,y) : \mathrm{Type}\right)$$

$$\begin{tikzcd} & \Pi_B C \arrow[d, two heads] \\ B \arrow[r, two heads] & A \end{tikzcd}$$

Dependent products $\longleftrightarrow$ "display maps exponentiate"

Identity types for display maps

The dependent identity type

$$(x\colon A), (y\colon A) \vdash ((x = y) : \mathrm{Type})$$

must be a display map

$$\begin{tikzcd} \mathrm{Id}_A \arrow[d, two heads] \\ A \times A \end{tikzcd}$$

Identity types for display maps

The reflexivity constructor

$$(x\colon A) \vdash (\mathsf{refl}(x) : (x = x))$$

must be a section

$$\begin{tikzcd} \Delta^*\mathsf{Id}_A \arrow[r] \arrow[d, two heads] & \mathsf{Id}_A \arrow[d, two heads] \\ A \arrow[r, "\Delta"'] \arrow[u, bend left] & A \times A \end{tikzcd}$$

or equivalently a lifting

$$\begin{tikzcd} & \mathsf{Id}_A \arrow[d, two heads] \\ A \arrow[ur, "\mathsf{refl}"] \arrow[r, "\Delta"'] & A \times A \end{tikzcd}$$

Identity types for display maps

The eliminator says given a dependent type with a section

$$\begin{tikzcd} \mathsf{refl}^*C \arrow[r] \arrow[d, two heads] & C \arrow[d, two heads] \\ A \arrow[r, "\mathsf{refl}"'] \arrow[u, bend left] & \mathsf{Id}_A \end{tikzcd}$$

there exists a compatible section

$$\begin{tikzcd} C \arrow[d, two heads] \\ \mathsf{Id}_A \arrow[u, bend left] \end{tikzcd}$$

In other words, we have the lifting property

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathsf{refl}"'] & C \arrow[d, two heads] \\ \mathsf{Id}_A \arrow[r, equal] \arrow[ur, dotted, "\exists"] & \mathsf{Id}_A \end{tikzcd}$$

Identity types for display maps

In fact, refl has the left lifting property w.r.t. all display maps.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathsf{refl}"'] & f^* C \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, two heads] \\
\mathsf{Id}_A \arrow[r, equal] \arrow[ur, dotted, "\exists"] & \mathsf{Id}_A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Conclusion

Identity types factor $\Delta\colon A \to A \times A$ as

$$A \xrightarrow{\mathsf{refl}} \mathsf{Id}_A \overset{q}{\twoheadrightarrow} A \times A$$

where q is a display map and refl lifts against all display maps.

Weak factorization systems

Definition

We say $j \boxslash f$ if any commutative square

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "j"'] & B \arrow[d, "f"] \\
Y \arrow[r] \arrow[ur, dotted, "\exists"] & A
\end{tikzcd}
$$

admits a (non-unique) diagonal filler.

- $\mathcal{J}^{\boxslash} = \{\, f \mid j \boxslash f \quad \forall j \in \mathcal{J} \,\}$
- ${}^{\boxslash}\mathcal{F} = \{\, j \mid j \boxslash f \quad \forall f \in \mathcal{F} \,\}$

Definition

A weak factorization system in a category is $(\mathcal{J}, \mathcal{F})$ such that

1. $\mathcal{J} = {}^{\boxslash}\mathcal{F}$ and $\mathcal{F} = \mathcal{J}^{\boxslash}$.
2. Every morphism factors as $f \circ j$ for some $f \in \mathcal{F}$ and $j \in \mathcal{J}$.

General factorizations

Theorem (Gambino–Garner)

In a display map category that models identity types, any morphism $g\colon A \to B$ factors as

$$A \xrightarrow{j} Ng \xrightarrow{f} B$$

where f is a display map, and j lifts against all display maps.

$$(y\colon B) \vdash \; Ng(y) :\equiv \mathsf{hfiber}(g, y) :\equiv \sum_{x\colon A} (g(x) = y)$$

is the type-theoretic mapping path space.

The identity type wfs

Corollary (Gambino-Garner)

In a type theory with identity types,

$$\left({}^{\boxslash}(\textit{display maps}), ({}^{\boxslash}(\textit{display maps}))^{\boxslash}\right)$$

is a weak factorization system.

This behaves very much like (acyclic cofibrations, fibrations):

- Dependent types are like fibrations (recall "transport").
- Every map in ${}^{\boxslash}$(display maps) is an equivalence; in fact, the inclusion of a deformation retract.

Modeling identity types

Conversely:

Theorem (Awodey–Warren, Garner–van den Berg)

In a display map category, if

$$\left({}^{\boxslash}(\textit{display maps}), ({}^{\boxslash}(\textit{display maps}))^{\boxslash}\right)$$

is a "pullback-stable" weak factorization system, then the category (almost) models identity types.*

identity types $\longleftrightarrow$ weak factorization systems

Model categories

Definition (Quillen)

A model category is a category $\mathbf{C}$ with limits and colimits and three classes of maps:

- $\mathcal{C}$ = cofibrations
- $\mathcal{F}$ = fibrations
- $\mathcal{W}$ = weak equivalences

such that

1. $\mathcal{W}$ has the 2-out-of-3 property.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.

Type-theoretic model categories

Corollary

Let $\mathcal{M}$ be a model category such that

1. $\mathcal{M}$ (as a category) is locally cartesian closed.
2. $\mathcal{M}$ is right proper.
3. The cofibrations are the monomorphisms.

Then $\mathcal{M}$ (almost*) models type theory with dependent sums, dependent products, and identity types.

$$\boxed{\text{Homotopy theory}} \xleftarrow[\text{(homotopy type) theory}]{} \boxed{\text{Type theory}}$$

Examples

- Simplicial sets with the Quillen model structure.
- Any injective model structure on simplicial presheaves.

Homotopy type theory in categories

$$(x\colon A) \vdash p\colon \mathrm{isProp}(B(x))$$

$$\iff (x\colon A), (u\colon B(x)), (v\colon B(x)) \vdash (p_{u,v}\colon (u = v))$$

$\iff$ The path object $P_A B$ has a section in $\mathcal{M}/A$

$\iff$ Any two maps into B are homotopic over A

$$(x\colon A) \vdash p\colon \mathrm{isContr}(B(x))$$

$$\iff (x\colon A) \vdash p\colon \mathrm{isProp}(B(x)) \times B(x)$$

$\iff$ Any two maps into B are homotopic over A

$\iff$ and $B \twoheadrightarrow A$ has a section

$\iff$ $B \twoheadrightarrow A$ is an acyclic fibration

Homotopy type theory in categories

For $f : A \to B$,

$$
\begin{aligned}
\vdash\ p : \mathsf{isEquiv}(f) &\iff \vdash \prod_{y : B} \mathsf{isContr}(\mathsf{hfiber}(f, y)) \\
&\iff (y : B) \vdash \mathsf{isContr}(\mathsf{hfiber}(f, y)) \\
&\iff \mathsf{hfiber}(f) \twoheadrightarrow A \text{ is an acyclic fibration} \\
&\iff f \text{ is a (weak) equivalence}
\end{aligned}
$$

(Recall hfiber is the factorization $A \to Nf \twoheadrightarrow B$ of f.)

Conclusion

Any theorem about "equivalences" that we can prove in type theory yields a conclusion about weak equivalences in appropriate model categories.

Coherence

Another Problem

Type theory is even stricter than 1-categories!

Recall that substitution is pullback.

$$
\begin{tikzcd}
f^*g^*A \arrow[r] \arrow[d] & g^*P \arrow[r] \arrow[d] & P \arrow[d] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}
$$

$a : A \vdash P(g(f(a)))$ $\qquad$ $b : B \vdash P(g(b))$ $\qquad$ $c : C \vdash P(c)$

Coherence

Another Problem

Type theory is even stricter than 1-categories!

Recall that substitution is pullback.

$$\begin{array}{ccc} (g \circ f)^* A & \longrightarrow & P \\ \downarrow & & \downarrow \\ A & \xrightarrow{g \circ f} & C \end{array}$$

$a \colon A \vdash P(g(f(a)))$ $\qquad\qquad$ $c \colon C \vdash P(c)$

But, of course, $f^* g^* P$ is only isomorphic to $(g \circ f)^* P$.

Coherence with a universe

There are several resolutions; perhaps the cleanest is:

Solution (Voevodsky)

Represent dependent types by their classifying maps into a universe object.

Now substitution is composition, which is strictly associative (in our model category):

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{P} U$$

$$A \xrightarrow{g \circ f} C \xrightarrow{P} U$$

We needed a universe object anyway, to model the type Type and prove univalence.

New problem

Need very strict models for universe objects.

Representing fibrations

(Following Kapulkin–Lumsdaine–Voevodsky)

Goal

A universe object in simplicial sets giving coherence and univalence.

Simplicial sets are a presheaf category, so there is a standard trick to build representing objects.

$$U_n \cong \mathrm{Hom}(\Delta^n, U) \simeq \{\text{fibrations over } \Delta^n\}$$

But $n \mapsto \{\text{fibrations over } \Delta^n\}$ is only a pseudofunctor; we need to rigidify it.

Well-ordered fibrations

A technical device (Voevodsky)

A well-ordered Kan fibration is a Kan fibration $p\colon E \to B$ together with, for every $x \in B_n$, a well-ordering on $p^{-1}(x) \subseteq E_n$.

Two well-ordered Kan fibrations are isomorphic in at most one way which preserves the orders.

Definition

$$U_n := \{X \twoheadrightarrow \Delta^n \text{ a well-ordered fibration}\} \Big/ _{\text{ordered } \cong}$$

$$\widetilde{U}_n := \left\{(X, x) \;\middle|\; X \twoheadrightarrow \Delta^n \text{ well-ordered fibration, } x \in X_n\right\} \Big/ _{\text{ordered } \cong}$$

(with some size restriction, to make them sets).

The universal Kan fibration

Theorem

The forgetful map $\widetilde{U} \to U$ is a Kan fibration.

Proof.

A map $E \to B$ is a Kan fibration if and only if every pullback

$$\begin{tikzcd} b^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ \Delta^n \arrow[r, "b"'] & B \end{tikzcd}$$

is such, since the horns $\Lambda^n_k \hookrightarrow \Delta^n$ have codomain Δ^n. □

Thus, of course, every pullback of $\widetilde{U} \to U$ is a Kan fibration.

The universal Kan fibration

Theorem

Every (small) Kan fibration $E \to B$ is some pullback of $\widetilde{U} \to U$:

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{U} \arrow[d] \\ B \arrow[r] & U \end{tikzcd}$$

Proof.

Choose a well-ordering on each fiber, and map $x \in B_n$ to the isomorphism class of the well-ordered fibration $b^*(E) \twoheadrightarrow \Delta^n$. □

It is essential that we have actual pullbacks here, not just homotopy pullbacks.

Type theory in the universe

Let the size-bound for U be inaccessible (a Grothendieck universe). Then small fibrations are closed under all categorical constructions.

Now we can interpret type theory with coherence, using morphisms into U for dependent types.

Example

A context

$$(x\colon A),\ (y\colon B(x)),\ (z\colon C(x,y))$$

becomes a sequence of fibrations together with classifying maps:

$$\begin{tikzcd}
C \arrow[rr, two heads] \arrow[dr] & & B \arrow[rr, two heads] \arrow[dl, "{[C]}"'] \arrow[dr] & & A \arrow[rr, two heads] \arrow[dl, "{[B]}"'] \arrow[dr] & & 1 \arrow[dl, "{[A]}"'] \\
& \widetilde{U} \twoheadrightarrow U & & \widetilde{U} \twoheadrightarrow U & & \widetilde{U} \twoheadrightarrow U &
\end{tikzcd}$$

in which each trapezoid is a pullback.

Strict cartesian products

Every type-theoretic operation can be done once over U, then implemented by composition.

Example (Cartesian product)

- Pull $\widetilde{U}$ back to $U \times U$ along the two projections π_1, π_2.
- Their fiber product over $U \times U$ admits a classifying map:

$$\begin{tikzcd}
(\pi_1^*\widetilde{U}) \times_{U\times U} (\pi_2^*\widetilde{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{U} \arrow[d] \\
U \times U \arrow[r, "{[\times]}"'] & U
\end{tikzcd}$$

- Define the product of $[A]\colon X \to U$ and $[B]\colon X \to U$ to be

$$X \xrightarrow{([A],[B])} U \times U \xrightarrow{[\times]} U$$

This has strict substitution.

Nested universes

Problem

So far the object U lives outside the type theory.
We want it inside, giving a universe type "Type" and univalence.

Solution

Let U' be a bigger universe. If U is U'-small and fibrant, then it has a classifying map:

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{U}' \arrow[d] \\ 1 \arrow[r, "u"'] & U' \end{tikzcd}$$

and the type theory defined using U' has a universe type u.

U is fibrant

Theorem

U is fibrant.

Outline of proof.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, "j"'] & U \\ \Delta^n \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

With hard work, we can extend $f^*\widetilde{U}$ to a fibration over Δ^n:

$$\begin{tikzcd} f^*\widetilde{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\ \Lambda^n_k \arrow[r, "j"'] & \Delta^n \end{tikzcd}$$

and extend the well-ordering of $f^*\widetilde{U}$ to P, yielding $g\colon \Delta^n \to U$ with $gj = f$ (and $g^*\widetilde{U} \cong P$). $\square$

Extending fibrations

Lemma

Any fibration $P \to \Lambda^n_k$ is the pullback of some fibration over Δ^n.

Proof.

- Let $P' \subseteq P$ be a minimal subfibration.
- There is a retraction $P \to P'$ that is an acyclic fibration.
- Since Λ^n_k is contractible, the minimal fibration $P' \to \Lambda^n_k$ is isomorphic to a trivial bundle $\Lambda^n_k \times F \to \Lambda^n_k$.

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_{j \times F} P \arrow[d] \\
P' \cong \Lambda^n_k \times F \arrow[r, "j \times F"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^n \times F \arrow[d] \\
\Lambda^n_k \arrow[r, "j"'] & \Delta^n
\end{tikzcd}
$$

□

Univalence

We want to show that $PU \to \mathrm{Eq}(U)$ is an equivalence:

$$
\begin{tikzcd}
U \arrow[r, "\sim"] \arrow[dr, "\Delta"'] \arrow[ddr, bend right, "\mathrm{id}"'] & PU \arrow[r, "?"] \arrow[d] & \mathrm{Eq}(U) \arrow[dl] \arrow[ddl, bend left, "?"] \\
& U \times U \arrow[d, "\pi_2"] & \\
& U &
\end{tikzcd}
$$

It suffices to show:

1. The composite $U \to \mathrm{Eq}(U)$ is an equivalence.
2. The projection $\mathrm{Eq}(U) \to U$ is an equivalence.
3. The projection $\mathrm{Eq}(U) \to U$ is an acyclic fibration.

Univalence

By representability, a commutative square with a lift

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, "i"'] & \mathrm{Eq}(U) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, red] & U \end{tikzcd}$$

corresponds to a diagram

$$\begin{tikzcd} E_1 \arrow[rr, red] \arrow[dr] \arrow[dd] & & \overline{E}_1 \arrow[dr, red] \arrow[dd, red] & \\ & E_2 \arrow[rr] \arrow[dl] & & \overline{E}_2 \arrow[dl] \\ \partial\Delta^n \arrow[rr, "i"'] & & \Delta^n & \end{tikzcd}$$

with $E_1 \to E_2$ an equivalence.and $\overline{E}_1 \to \overline{E}_2$ equivalences.

Univalence

$$\begin{tikzcd} E_1 \arrow[rr, red] \arrow[dr] \arrow[dd] & & \overline{E}_1 \arrow[rr, red] \arrow[dr, red] & & \Pi_i(E_1) \arrow[dr, red] & \\ & E_2 \arrow[rr, "e_2"'] \arrow[dl] & & \overline{E}_2 \arrow[dl] \arrow[rr, red] & & \Pi_i(E_2) \\ \partial\Delta^n \arrow[rr, "i"'] & & \Delta^n & & & \end{tikzcd}$$

- By factorization, consider separately the cases when $E_1 \to E_2$ is (1) an acyclic fibration or (2) an acyclic cofibration.
- (1) $\overline{E}_1 \to \overline{E}_2$ is an acyclic fibration (Π_i preserves such).
- (2) $\overline{E}_1$ is a deformation retract of $\overline{E}_2$.

$(\infty, 1)$-toposes

Definition

An $(\infty, 1)$-topos is an $(\infty, 1)$-category that is a left-exact localization of an $(\infty, 1)$-presheaf category.

Examples

- ∞-groupoids (plays the role of the 1-topos Set)
- Parametrized homotopy theory over any space X
- G-equivariant homotopy theory for any group G
- ∞-sheaves/stacks on any space
- "Smooth ∞-groupoids" (or "algebraic" etc.)

Univalence in categories

Definition (Rezk)

An object classifier in an $(\infty, 1)$-category $\mathcal{C}$ is a morphism $\widetilde{U} \to U$ such that pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{U} \arrow[d] \\ A \arrow[r] & U \end{tikzcd}$$

induces an equivalence of ∞-groupoids

$$\mathrm{Hom}(A, U) \xrightarrow{\sim} \mathrm{Core}(\mathcal{C}/A)_{\mathrm{small}}$$

("Core" is the maximal sub-∞-groupoid.)

$(\infty, 1)$-toposes

Theorem (Rezk)

An $(\infty, 1)$-category $\mathcal{C}$ is an $(\infty, 1)$-topos if and only if

1. *$\mathcal{C}$ is locally presentable.*
2. *$\mathcal{C}$ is locally cartesian closed.*
3. *κ-compact objects have object classifiers for $\kappa \gg 0$.*

Corollary

If a combinatorial model category $\mathcal{M}$ interprets dependent type theory as before (i.e. it is locally cartesian closed, right proper, and the cofibrations are the monomorphisms), and contains universes for κ-compact objects that satisfy the univalence axiom, then the $(\infty, 1)$-category that it presents is an $(\infty, 1)$-topos.

$(\infty, 1)$-toposes

Conjecture

Every $(\infty, 1)$-topos can be presented by a model category which interprets dependent type theory with the univalence axiom.

Homotopy type theory is the internal logic of $(\infty, 1)$-toposes.

If this is true, then anything we prove in homotopy type theory (which we can also verify with a computer) will automatically be true internally to any $(\infty, 1)$-topos. The "constructive core" of homotopy theory should be provable in this way, in a uniform way for "all homotopy theories".

Status of the conjecture

$$
\begin{tikzcd}
\infty\mathit{Gpd} \arrow[r, dashed] \arrow[d, "\checkmark"] & (\infty,1)\text{-presheaves} \arrow[r, "\checkmark"] & (\infty,1)\text{-toposes} \\
\text{inverse } (\infty,1)\text{-presheaves} \arrow[ur, dashed, "\checkmark?"'] & &
\end{tikzcd}
$$