

Categorical models of homotopy type theory

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Homotopy type theory in higher categories

Recall:

homotopy type theory	\longleftrightarrow	$(\infty, 1)$ -categories
$\times, +$ types	\longleftrightarrow	products, coproducts
equality types $(x = y)$	\longleftrightarrow	diagonals
\prod types	\longleftrightarrow	local cartesian closure
univalent universe \mathbf{Type}	\longleftrightarrow	object classifier

Two kinds of equality

Problem

Type theory is **stricter** than $(\infty, 1)$ -categories.

In type theory, we have two kinds of “equality”:

- ① Equality witnessed by inhabitants of **equality types** (= paths).
- ② **Computational** equality: $(\lambda x. b)(a)$ evaluates to $b[a/x]$.

These play different roles: **type checking** depends on **computational** equality.

- if a evaluates to b , and $c : C(a)$, then also $c : C(b)$.
 - In particular, if a evaluates to b , then $\text{refl}_b : (a = b)$.
- if $p : (a = b)$ and $c : C(a)$, then only $\text{transport}(p, c) : C(b)$.

Two kinds of equality

But computational equality is also **stricter**.

Example

Composition is computationally strictly associative.

$$g \circ f := \lambda x. g(f(x))$$

$$h \circ (g \circ f) = \lambda x. h\left((\lambda x. g(f(x)))(x)\right) \rightsquigarrow \lambda x. h(g(f(x)))$$

$$(h \circ g) \circ f = \lambda x. \left(\lambda y. h(g(y))\right)(f(x)) \rightsquigarrow \lambda x. h(g(f(x)))$$

- This is the sort of issue that homotopy theorists are intimately familiar with!
- We need a model for $(\infty, 1)$ -categories with (at least) a strictly associative composition law.

Display map categories

Forget everything you know about homotopy theory; let's see how the type theorists come at it.

Definition

A **display map category** is a category with

- A terminal object.
 - A subclass of its morphisms called the **display maps**, denoted $P \twoheadrightarrow A$ or $P \rightarrow A$.
 - Any pullback of a display map exists and is a display map.
-
- A display map $P \twoheadrightarrow A$ is a type dependent on A .
 - A display map $A \twoheadrightarrow 1$ is a plain type (dependent on nothing).
 - Pullback is substitution.

Dependent sums of display maps

$$(x : A) \vdash (B(x) : \text{Type})$$

If the types $B(x)$ are the fibers of $B \twoheadrightarrow A$, their dependent sum $\sum_{x:A} B(x)$ should be the object B .

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ & & \Downarrow \\ (x : A) \vdash (B(x) : \text{Type}) & & A \\ & & \downarrow \\ & & \Downarrow \\ & & 1 \\ & & \\ & & B \\ & & \downarrow \\ & & \Downarrow \\ \vdash \left(\sum_{x:A} B(x) : \text{Type} \right) & & 1 \end{array}$$

Dependent sums in context

More generally:

$$(x : A), (y : B(x)) \vdash (C(x, y) : \text{Type})$$

$$\begin{array}{c} C \\ \downarrow \\ B \\ \downarrow \\ A \end{array}$$

$$(x : A) \vdash \left(\sum_{y : B(x)} C(x, y) : \text{Type} \right)$$

$$\begin{array}{c} C \\ \downarrow \\ A \end{array}$$

Dependent sums \longleftrightarrow display maps compose

Aside: adjoints to pullback

- In a category \mathcal{C} , if pullbacks along $f : A \rightarrow B$ exist, then the functor

$$f^* : \mathcal{C}/B \longrightarrow \mathcal{C}/A$$

has a left adjoint Σ_f given by composition with f .

- If f is a display map and display maps compose, then Σ_f restricts to a functor

$$(\mathcal{C}/A)_{\text{disp}} \longrightarrow (\mathcal{C}/B)_{\text{disp}}$$

implementing dependent sums.

- A right adjoint to f^* , if one exists, is an “object of sections”. \mathcal{C} is **locally cartesian closed** iff all such right adjoints Π_f exist.

Dependent products of display maps

$$(x : A), (y : B(x)) \vdash (C(x, y) : \text{Type})$$

$$\begin{array}{c} C \\ \downarrow \\ B \multimap A \end{array}$$

$$(x : A) \vdash \left(\prod_{y : B(x)} C(x, y) : \text{Type} \right)$$

$$\begin{array}{c} \Pi_B C \\ \downarrow \\ B \multimap A \end{array}$$

Dependent products \longleftrightarrow “display maps exponentiate”

Identity types for display maps

The dependent **identity type**

$$(x : A), (y : A) \vdash ((x = y) : \text{Type})$$

must be a display map

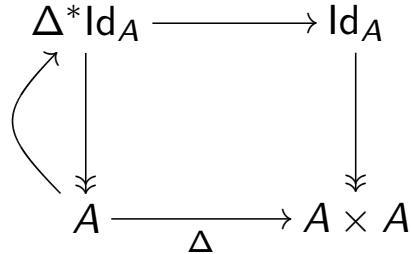
$$\begin{array}{c} \text{Id}_A \\ \downarrow \\ A \times A \end{array}$$

Identity types for display maps

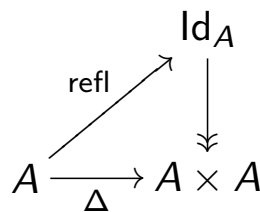
The **reflexivity constructor**

$$(x : A) \vdash (\text{refl}(x) : (x = x))$$

must be a section

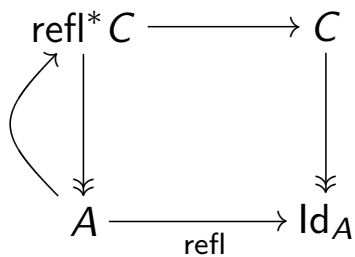


or equivalently a lifting

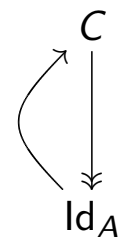


Identity types for display maps

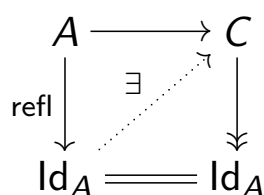
The **eliminator** says given a dependent type with a section



there exists
a compatible
section



In other words, we have the lifting property



Identity types for display maps

In fact, refl has the left lifting property w.r.t. **all** display maps.

$$\begin{array}{ccccc}
 A & \longrightarrow & f^*C & \longrightarrow & C \\
 \downarrow \text{refl} & & \swarrow \exists & \downarrow \lrcorner & \downarrow \\
 \text{Id}_A & \xlongequal{\quad} & \text{Id}_A & \xrightarrow{f} & B
 \end{array}$$

Conclusion

Identity types factor $\Delta: A \rightarrow A \times A$ as

$$A \xrightarrow{\text{refl}} \text{Id}_A \xrightarrow{q} A \times A$$

where q is a display map and refl lifts against all display maps.

Weak factorization systems

Definition

We say $j \boxtimes f$ if any commutative square

$$\begin{array}{ccc}
 X & \longrightarrow & B \\
 j \downarrow & \swarrow \exists & \downarrow f \\
 Y & \longrightarrow & A
 \end{array}$$

admits a (non-unique) diagonal filler.

- $\mathcal{J}^{\boxtimes} = \{ f \mid j \boxtimes f \ \forall j \in \mathcal{J} \}$
- $\boxtimes \mathcal{F} = \{ j \mid j \boxtimes f \ \forall f \in \mathcal{F} \}$

Definition

A **weak factorization system** in a category is $(\mathcal{J}, \mathcal{F})$ such that

- ① $\mathcal{J} = \boxtimes \mathcal{F}$ and $\mathcal{F} = \mathcal{J}^{\boxtimes}$.
- ② Every morphism factors as $f \circ j$ for some $f \in \mathcal{F}$ and $j \in \mathcal{J}$.

General factorizations

Theorem (Gambino–Garner)

In a display map category that models identity types, any morphism $g: A \rightarrow B$ factors as

$$A \xrightarrow{j} Ng \xrightarrow{f} \twoheadrightarrow B$$

where f is a display map, and j lifts against all display maps.

$$(y: B) \vdash Ng(y) := \text{hfiber}(g, y) := \sum_{x: A} (g(x) = y)$$

is the type-theoretic **mapping path space**.

The identity type wfs

Corollary (Gambino–Garner)

In a type theory with identity types,

$$\left(\square(\text{display maps}), (\square(\text{display maps}))^\square \right)$$

is a weak factorization system.

This behaves very much like (acyclic cofibrations, fibrations):

- Dependent types are like fibrations (recall “transport”).
- Every map in $\square(\text{display maps})$ is an equivalence; in fact, the inclusion of a deformation retract.

Modeling identity types

Conversely:

Theorem (Awodey–Warren, Garner–van den Berg)

In a display map category, if

$$\left(\square(\text{display maps}), (\square(\text{display maps}))^\square \right)$$

is a “pullback-stable” weak factorization system, then the category (almost*) models identity types.

identity types \longleftrightarrow weak factorization systems

Model categories

Definition (Quillen)

A **model category** is a category \mathbf{C} with limits and colimits and three classes of maps:

- \mathcal{C} = cofibrations
- \mathcal{F} = fibrations
- \mathcal{W} = weak equivalences

such that

- ① \mathcal{W} has the 2-out-of-3 property.
- ② $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.

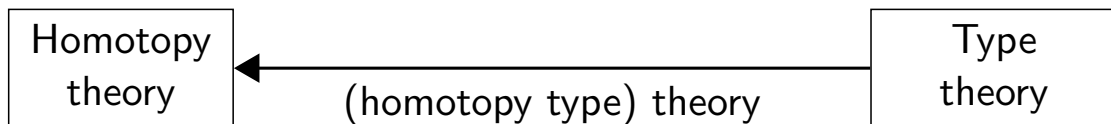
Type-theoretic model categories

Corollary

Let \mathcal{M} be a model category such that

- ① \mathcal{M} (as a category) is locally cartesian closed.
- ② \mathcal{M} is right proper.
- ③ The cofibrations are the monomorphisms.

Then \mathcal{M} (almost*) models type theory with dependent sums, dependent products, and identity types.



Examples

- Simplicial sets with the **Quillen** model structure.
- Any injective model structure on simplicial presheaves.

Homotopy type theory in categories

$(x: A) \vdash p: \text{isProp}(B(x))$

$\iff (x: A), (u: B(x)), (v: B(x)) \vdash (p_{u,v}: (u = v))$

\iff The path object $P_A B$ has a section in \mathcal{M}/A

\iff Any two maps into B are homotopic over A

$(x: A) \vdash p: \text{isContr}(B(x))$

$\iff (x: A) \vdash p: \text{isProp}(B(x)) \times B(x)$

\iff Any two maps into B are homotopic over A

\iff and $B \twoheadrightarrow A$ has a section

$\iff B \twoheadrightarrow A$ is an acyclic fibration

Homotopy type theory in categories

For $f: A \rightarrow B$,

$$\begin{aligned} \vdash p: \text{isEquiv}(f) &\iff \vdash \prod_{y: B} \text{isContr}(\text{hfiber}(f, y)) \\ &\iff (y: B) \vdash \text{isContr}(\text{hfiber}(f, y)) \\ &\iff \text{hfiber}(f) \twoheadrightarrow A \text{ is an acyclic fibration} \\ &\iff f \text{ is a (weak) equivalence} \end{aligned}$$

(Recall hfiber is the factorization $A \rightarrow Nf \twoheadrightarrow B$ of f .)

Conclusion

Any theorem about “equivalences” that we can prove in type theory yields a conclusion about weak equivalences in appropriate model categories.

Coherence

Another Problem

Type theory is even stricter than 1-categories!

Recall that **substitution** is **pullback**.

$$\begin{array}{ccccc} f^*g^*A & \longrightarrow & g^*P & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

$$a: A \vdash P(g(f(a))) \quad b: B \vdash P(g(b)) \quad c: C \vdash P(c)$$

Coherence

Another Problem

Type theory is even stricter than 1-categories!

Recall that **substitution** is **pullback**.

$$\begin{array}{ccc} (g \circ f)^* A & \xrightarrow{\quad} & P \\ \downarrow & & \downarrow \\ A & \xrightarrow{g \circ f} & C \end{array}$$

$$a: A \vdash P(g(f(a)))$$

$$c: C \vdash P(c)$$

But, of course, f^*g^*P is only **isomorphic** to $(g \circ f)^*P$.

Coherence with a universe

There are several resolutions; perhaps the cleanest is:

Solution (Voevodsky)

Represent dependent types by their **classifying maps** into a universe object.

Now substitution is **composition**, which is strictly associative (in our model category):

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{P} & U \\ & & & & \xrightarrow{g \circ f} & & \\ A & \xrightarrow{g \circ f} & C & \xrightarrow{P} & U & & \end{array}$$

We needed a universe object anyway, to model the type `Type` and prove univalence.

New problem

Need **very strict models** for universe objects.

Representing fibrations

(Following Kapulkin–Lumsdaine–Voevodsky)

Goal

A universe object in simplicial sets giving **coherence** and **univalence**.

Simplicial sets are a presheaf category, so there is a standard trick to build representing objects.

$$U_n \cong \text{Hom}(\Delta^n, U) \simeq \{\text{fibrations over } \Delta^n\}$$

But $n \mapsto \{\text{fibrations over } \Delta^n\}$ is only a pseudofunctor; we need to **rigidify** it.

Well-ordered fibrations

A technical device (Voevodsky)

A **well-ordered Kan fibration** is a Kan fibration $p: E \rightarrow B$ together with, for every $x \in B_n$, a well-ordering on $p^{-1}(x) \subseteq E_n$.

Two well-ordered Kan fibrations are isomorphic in **at most one way** which preserves the orders.

Definition

$$U_n := \{X \twoheadrightarrow \Delta^n \text{ a well-ordered fibration}\} /_{\text{ordered}} \cong$$

$$\tilde{U}_n := \{(X, x) \mid X \twoheadrightarrow \Delta^n \text{ well-ordered fibration, } x \in X_n\} /_{\text{ordered}} \cong$$

(with some size restriction, to make them sets).

The universal Kan fibration

Theorem

The forgetful map $\tilde{U} \rightarrow U$ is a Kan fibration.

Proof.

A map $E \rightarrow B$ is a Kan fibration if and only if every pullback

$$\begin{array}{ccc} b^*E & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ \Delta^n & \xrightarrow{b} & B \end{array}$$

is such, since the horns $\Lambda_k^n \hookrightarrow \Delta^n$ have codomain Δ^n . □

Thus, of course, every pullback of $\tilde{U} \rightarrow U$ is a Kan fibration.

The universal Kan fibration

Theorem

Every (small) Kan fibration $E \rightarrow B$ is some pullback of $\tilde{U} \rightarrow U$:

$$\begin{array}{ccc} E & \longrightarrow & \tilde{U} \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & U \end{array}$$

Proof.

Choose a well-ordering on each fiber, and map $x \in B_n$ to the isomorphism class of the well-ordered fibration $b^*(E) \rightarrow \Delta^n$. □

It is essential that we have **actual** pullbacks here, not just homotopy pullbacks.

Type theory in the universe

Let the size-bound for U be **inaccessible** (a Grothendieck universe). Then small fibrations are closed under all categorical constructions.

Now we can interpret type theory **with coherence**, using morphisms into U for dependent types.

Example

A context

$$(x: A), (y: B(x)), (z: C(x, y))$$

becomes a sequence of fibrations together with classifying maps:

$$\begin{array}{ccccccc}
 C & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A & \xrightarrow{\quad} & 1 \\
 & \searrow & \swarrow & & \searrow & & \swarrow \\
 & & \tilde{U} & \twoheadrightarrow & U & & \\
 & & \swarrow & & \searrow & & \\
 & & U & & U & & \\
 & & \twoheadrightarrow & & \twoheadrightarrow & & \\
 & & U & & U & &
 \end{array}$$

[C] [B] [A]

in which each trapezoid is a pullback.

Strict cartesian products

Every type-theoretic operation can be done **once** over U , then implemented by composition.

Example (Cartesian product)

- Pull \tilde{U} back to $U \times U$ along the two projections π_1, π_2 .
- Their fiber product over $U \times U$ admits a classifying map:

$$\begin{array}{ccc}
 (\pi_1^* \tilde{U}) \times_{U \times U} (\pi_2^* \tilde{U}) & \longrightarrow & \tilde{U} \\
 \downarrow \lrcorner & & \downarrow \\
 U \times U & \xrightarrow{[\times]} & U
 \end{array}$$

- Define the product of $[A]: X \rightarrow U$ and $[B]: X \rightarrow U$ to be

$$X \xrightarrow{([A],[B])} U \times U \xrightarrow{[\times]} U$$

This has strict substitution.

Nested universes

Problem

So far the object U lives **outside** the type theory.
We want it **inside**, giving a universe type “Type” and univalence.

Solution

Let U' be a bigger universe. If U is U' -small **and fibrant**, then it has a classifying map:

$$\begin{array}{ccc} U & \longrightarrow & \tilde{U}' \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow{u} & U' \end{array}$$

and the type theory defined using U' has a universe type u .

U is fibrant

Theorem

U is fibrant.

Outline of proof.

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{f} & U \\ j \downarrow & \nearrow ? & \\ \Delta^n & & \end{array}$$

With **hard work**, we can extend $f^*\tilde{U}$ to a fibration over Δ^n :

$$\begin{array}{ccc} f^*\tilde{U} & \longrightarrow & P \\ \downarrow \lrcorner & & \downarrow \\ \Lambda_k^n & \xrightarrow{j} & \Delta^n \end{array}$$

and extend the well-ordering of $f^*\tilde{U}$ to P , yielding $g: \Delta^n \rightarrow U$ with $gj = f$ (and $g^*\tilde{U} \cong P$). □

Extending fibrations

Lemma

Any fibration $P \rightarrow \Lambda_k^n$ is the pullback of some fibration over Δ^n .

Proof.

- Let $P' \subseteq P$ be a **minimal subfibration**.
- There is a retraction $P \rightarrow P'$ that is an acyclic fibration.
- Since Λ_k^n is contractible, the minimal fibration $P' \rightarrow \Lambda_k^n$ is **isomorphic** to a trivial bundle $\Lambda_k^n \times F \rightarrow \Lambda_k^n$.

$$\begin{array}{ccc}
 P & \xrightarrow{\quad} & \Pi_{j \times F} P \\
 \downarrow \lrcorner & & \downarrow \\
 P' \cong \Lambda_k^n \times F & \xrightarrow{j \times F} & \Delta^n \times F \\
 \downarrow \lrcorner & & \downarrow \\
 \Lambda_k^n & \xrightarrow{j} & \Delta^n
 \end{array}$$

□

Univalence

We want to show that $PU \rightarrow \text{Eq}(U)$ is an equivalence:

$$\begin{array}{ccccc}
 U & \xrightarrow{\sim} & PU & \xrightarrow{?} & \text{Eq}(U) \\
 & \searrow & \downarrow & \swarrow & \downarrow \\
 & & U \times U & & U \\
 & \searrow \text{id} & \downarrow \pi_2 & \swarrow ? & \\
 & & U & &
 \end{array}$$

It suffices to show:

- 1 The composite $U \rightarrow \text{Eq}(U)$ is an equivalence.
- 2 The projection $\text{Eq}(U) \rightarrow U$ is an equivalence.
- 3 The projection $\text{Eq}(U) \rightarrow U$ is an acyclic fibration.

Univalence

By representability, a commutative square **with a lift**

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \text{Eq}(U) \\ i \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & U \end{array}$$

corresponds to a diagram

$$\begin{array}{ccccc} E_1 & \xrightarrow{\quad} & \bar{E}_1 & & \\ & \searrow \lrcorner & & \searrow & \\ & & E_2 & \xrightarrow{\quad} & \bar{E}_2 \\ & & & \searrow & \\ & & \partial\Delta^n & \xrightarrow{i} & \Delta^n \end{array}$$

with $E_1 \rightarrow E_2$ an equivalence, and $\bar{E}_1 \rightarrow \bar{E}_2$ equivalences.

Univalence

$$\begin{array}{ccccccc} E_1 & \xrightarrow{\quad} & \bar{E}_1 & \xrightarrow{\quad} & \Pi_i(E_1) & & \\ & \searrow & & \searrow \lrcorner & & \searrow & \\ & & E_2 & \xrightarrow{e_2} & \bar{E}_2 & \xrightarrow{\quad} & \Pi_i(E_2) \\ & & & \searrow & & & \\ & & \partial\Delta^n & \xrightarrow{i} & \Delta^n & & \end{array}$$

- By factorization, consider separately the cases when $E_1 \rightarrow E_2$ is (1) an acyclic fibration or (2) an acyclic cofibration.
- (1) $\bar{E}_1 \rightarrow \bar{E}_2$ is an acyclic fibration (Π_i preserves such).
- (2) \bar{E}_1 is a deformation retract of \bar{E}_2 .

$(\infty, 1)$ -toposes

Definition

An $(\infty, 1)$ -topos is an $(\infty, 1)$ -category that is a left-exact localization of an $(\infty, 1)$ -presheaf category.

Examples

- ∞ -groupoids (plays the role of the 1-topos Set)
- Parametrized homotopy theory over any space X
- G -equivariant homotopy theory for any group G
- ∞ -sheaves/stacks on any space
- “Smooth ∞ -groupoids” (or “algebraic” etc.)

Univalence in categories

Definition (Rezk)

An **object classifier** in an $(\infty, 1)$ -category \mathcal{C} is a morphism $\tilde{U} \rightarrow U$ such that pullback

$$\begin{array}{ccc} B & \longrightarrow & \tilde{U} \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & U \end{array}$$

induces an equivalence of ∞ -groupoids

$$\mathrm{Hom}(A, U) \xrightarrow{\simeq} \mathrm{Core}(\mathcal{C}/A)_{\mathrm{small}}$$

(“Core” is the maximal sub- ∞ -groupoid.)

$(\infty, 1)$ -toposes

Theorem (Rezk)

An $(\infty, 1)$ -category \mathcal{C} is an $(\infty, 1)$ -topos if and only if

- 1 \mathcal{C} is locally presentable.
- 2 \mathcal{C} is locally cartesian closed.
- 3 κ -compact objects have object classifiers for $\kappa \gg 0$.

Corollary

If a combinatorial model category \mathcal{M} interprets dependent type theory as before (i.e. it is locally cartesian closed, right proper, and the cofibrations are the monomorphisms), and contains universes for κ -compact objects that satisfy the univalence axiom, then the $(\infty, 1)$ -category that it presents is an $(\infty, 1)$ -topos.

$(\infty, 1)$ -toposes

Conjecture

Every $(\infty, 1)$ -topos can be presented by a model category which interprets dependent type theory with the univalence axiom.

Homotopy type theory is the internal logic of $(\infty, 1)$ -toposes.

If this is true, then anything we prove in homotopy type theory (which we can also verify with a computer) will automatically be true internally to any $(\infty, 1)$ -topos. The “constructive core” of homotopy theory should be provable in this way, in a uniform way for “all homotopy theories”.

Status of the conjecture

