

Inductive and higher inductive types

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Homotopy invariance

Question

Suppose two model categories \mathcal{M} , \mathcal{N} present the same $(\infty, 1)$ -category \mathcal{C} . Do they have the same internal type theory?

- ① All type-theoretic operations are homotopy invariant (represent well-defined $(\infty, 1)$ -categorical operations).
- ② Therefore, any type-theoretic **construction** performed on equivalent data in \mathcal{M} and \mathcal{N} yields equivalent results.
- ③ All type-theoretic **data** is terms in (dependent) types, i.e. sections of fibrations. If all objects in \mathcal{M} and \mathcal{N} are cofibrant, any “section” in \mathcal{C} can be represented in both \mathcal{M} and \mathcal{N} .
- ④ The only trouble is with asserting computational equalities, e.g. “let G be a group with computationally associative multiplication”. If we stick with properties that can be **expressed in the type theory**, we are fine.

Positive types

Recall: **positive types** are characterized by their introduction rules.

In fact, any choice of introduction rule(s) determines a positive type in an algorithmic way.

- The derived eliminator literally does a case analysis on the introduction rules.
- We call these introduction rules **constructors**.

Positive types

Example (Coproduct types)

- **Introduction:** $\text{inl} : A \rightarrow A + B$ and $\text{inr} : B \rightarrow A + B$
- **Elimination:** If $(x : A) \vdash (c_A : C(\text{inl}(x)))$ and $(y : B) \vdash (c_B : C(\text{inr}(y)))$, then for $p : A + B$ we have $\text{case}(p, c_A, c_B) : C(p)$.

Example (Empty type)

- **Introduction:**
- **Elimination:** If (nothing), then for $p : \emptyset$ we have $\text{abort}(p) : C(p)$.

The natural numbers

The **natural numbers** are a positive type.

- ① **Formation:** There is a type \mathbb{N} .
- ② **Introduction:** $0 : \mathbb{N}$, and $(x : \mathbb{N}) \vdash (s(x) : \mathbb{N})$.

A new feature: the **input** of the constructor “s” involves something of the type \mathbb{N} being defined!

We intend, of course, that all elements of \mathbb{N} are generated by *successively* applying constructors.

$$0, s(0), s(s(0)), s(s(s(0))), \dots$$

The natural numbers

- ① **Formation:** There is a type \mathbb{N} .
- ② **Introduction:** $0 : \mathbb{N}$, and $(x : \mathbb{N}) \vdash (s(x) : \mathbb{N})$.
- ③ **Elimination?** If $c_0 : C(0)$ and $(x : \mathbb{N}) \vdash (c_s : C(s(x)))$, then for $p : \mathbb{N}$ we have $\text{match}(p, c_0, c_s) : C(p)$.

But this is not much good; we need to **recurse**.

- ③ **Elimination:** If $c_0 : C(0)$ and

$$(x : \mathbb{N}), (r : C(x)) \vdash (c_s : C(s(x)))$$

then for $p : \mathbb{N}$ we have $\text{rec}(p, c_0, c_s) : C(p)$.

The variable r represents the result of the recursive call at x , to be used the computation c_s of the value at $s(x)$.

Example: Addition

We define addition by recursion on the first input.

$$\begin{aligned}\text{plus}(0, m) &:= m \\ \text{plus}(s(n), m) &:= s(\text{plus}(n, m))\end{aligned}$$

In terms of the rec eliminator, this is

$$(n: \mathbb{N}), (m: \mathbb{N}) \vdash \text{plus}(n, m) := \text{rec}(n, m, s(r))$$

- When $n = 0$, the result is m .
- When n is a successor $s(x)$, the result is $s(r)$.
(As before, r is the result of the recursive call at x .)

The natural numbers

- ① **Formation:** There is a type \mathbb{N} .
- ② **Introduction:** $0: \mathbb{N}$, and $(x: \mathbb{N}) \vdash (s(x): \mathbb{N})$.
- ③ **Elimination:** If $c_0: C(0)$ and

$$(x: \mathbb{N}), (r: C(x)) \vdash (c_s: C(s(x)))$$

then for $p: \mathbb{N}$ we have $\text{rec}(p, c_0, c_s): C(p)$.

- ④ **Computation:**
 - $\text{rec}(0, c_0, c_s)$ computes to c_0 .
 - $\text{rec}(s(n), c_0, c_s)$ computes to c_s with n substituted for x and $\text{rec}(n, c_0, c_s)$ substituted for r .

Computing an addition

$$\begin{aligned}\text{plus}(ss0, sss0) &:= \text{rec}(ss0, sss0, s(r)) \\ &\rightsquigarrow s(\text{rec}(s0, sss0, s(r))) \\ &\rightsquigarrow s(s(\text{rec}(0, sss0, s(r)))) \\ &\rightsquigarrow s(s(sss0)) = sssss0\end{aligned}$$

Other recursive inductive types

Generalized positive types of this sort are called **inductive types**.

Example (Lists)

For any type A , there is a type $\text{List}(A)$, with constructors

$$\begin{aligned}&\vdash \text{nil} : \text{List}(A) \\ (a : A), (\ell : \text{List}(A)) &\vdash (\text{cons}(a, \ell) : \text{List}(A))\end{aligned}$$

Functional programming is built on defining functions by recursion over inductive datatypes.

$$\begin{aligned}\text{length}(\text{nil}) &:= 0 \\ \text{length}(\text{cons}(a, \ell)) &:= s(\text{length}(\ell))\end{aligned}$$

This is defined using the eliminator for $\text{List}(A)$.

Proof by induction

③ If $c_0: C(0)$ and

$$(x: \mathbb{N}), (r: C(x)) \vdash (c_s: C(s(x)))$$

then for $p: \mathbb{N}$ we have $\text{rec}(p, c_0, c_s) : C(p)$.

When C is a **predicate**, this is just **proof by induction**.

types	\longleftrightarrow	propositions
programming	\longleftrightarrow	proving
recursion	\longleftrightarrow	induction

Conclusion

Proof by induction is not something special about the natural numbers; it applies to any inductive type.

Recursively defined types

We can define dependent types as Type-valued recursive functions.

Theorem

$0 \neq 1$.

Proof.

Define $P: \mathbb{N} \rightarrow \text{Type}$ by “recursion”:

$$\begin{aligned} P(0) &:= \mathbf{1} \\ P(s(n)) &:= \emptyset \end{aligned}$$

- Suppose $p: (0 = 1)$.
- Since $\star : P(0)$, we have $\text{trans}(p, \star) : P(1) \equiv \emptyset$.
- Thus, $\lambda p. \text{trans}(p, tt) : ((0 = 1) \rightarrow \emptyset) \equiv \neg(0 = 1)$.

□

Example: Truncation

Definition

An ∞ -groupoid is **n -truncated** if it has no nontrivial k -morphisms for any $k > n$.

- h-sets are 0-truncated.
- A is $(n + 1)$ -truncated \iff each $(x = y)$ is n -truncated.
- A is an h-set \iff each $(x = y)$ is an h-prop.
Thus, it makes sense to call h-props “ **(-1) -truncated**”.
- A is an h-prop \iff each $(x = y)$ is contractible.
Thus, we call contractible spaces “ **(-2) -truncated**”.
- After this, it’s “turtles all the way down”: (-3) -truncated is the same as (-2) -truncated.
- (Voevodsky) **h-level n** means $(n - 2)$ -truncated.

$$\text{isHlevel}(0, A) := \text{isContr}(A)$$

$$\text{isHlevel}(s(n), A) := \prod_{x: A} \prod_{y: A} \text{isHlevel}(n, (x = y))$$

Inductive families

We can define **dependent** types inductively as well.

Example (Vectors)

For any A there is a dependent type $\text{Vec}(A): \mathbb{N} \rightarrow \text{Type}$, with constructors

$$\begin{aligned} & \vdash \text{nil} : \text{Vec}(A, 0) \\ (a: A), (n: \mathbb{N}), (\ell: \text{Vec}(A, n)) & \vdash (\text{cons}(a, \ell) : \text{Vec}(A, s(n))) \end{aligned}$$

(We build the length of a list into its type.)

Example (Equality!)

For any A there is a dependent type $\text{Eq}_A: A \times A \rightarrow \text{Type}$, with constructor

$$(a: A) \vdash (\text{refl}_a : \text{Eq}_A(a, a))$$

Natural numbers objects

The positive type \mathbb{N} should have a left universal property.

Definition

A **natural numbers object** is N with $0: 1 \rightarrow N$, $s: N \rightarrow N$, s.t.

- For any object X with $0_X: 1 \rightarrow X$ and $s_X: X \rightarrow X$, there is a unique $r: N \rightarrow X$ such that

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\ & \searrow^{0_X} & \downarrow r & & \downarrow r \\ & & X & \xrightarrow{s_X} & X \end{array}$$

= an initial object in the category of triples $(X, 1 \rightarrow X, X \rightarrow X)$.

natural numbers type \mathbb{N} \longleftrightarrow natural numbers object

Algebras for endofunctors

Let F be a functor from a category to itself.

Definition

An **F -algebra** is an object X with a morphism $x: F(X) \rightarrow X$.

An **F -algebra map** is a map $f: X \rightarrow Y$ such that

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ x \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

An **initial F -algebra** is an initial object in the category of F -algebras and F -algebra maps.

Inductive types and endofunctors

inductive types \longleftrightarrow initial algebras for endofunctors

inductive type		endofunctor
\mathbb{N}	\longleftrightarrow	$F(X) := 1 + X$
List(A)	\longleftrightarrow	$F(X) := 1 + (A \times X)$
$A + B$	\longleftrightarrow	$F(X) := A + B$ (a <i>constant</i> endofunctor)

The eliminator directly asserts only weak initiality, but using the **dependent** eliminator one can prove:

Theorem (Awodey–Gambino–Sojakova)

Any inductive type W is a **homotopy initial** F -algebra: the space of F -algebra maps $W \rightarrow X$ is contractible.

Constructing initial algebras

We also have:

Theorem

If F is an accessible endofunctor of a locally presentable category, then there exists an initial F -algebra.

Sketch of proof.

Take the colimit of the transfinite sequence

$$\emptyset \rightarrow F(\emptyset) \rightarrow F(F(\emptyset)) \rightarrow \dots$$

□

Higher inductive types

Idea

- **Inductive types** are a good way to build **sets**: we specify the elements of a set by giving constructors.
- To build an **space** (or ∞ -groupoid), we need to specify not only elements, but paths and higher paths.
- The iterative construction of initial algebras looks a lot like the small object argument.
- Is there an analogous notion of **higher inductive type** that describes more general cell complexes?

Example

The circle S^1 should be inductively defined by two constructors

$$\text{base} : S^1 \quad \text{and} \quad \text{loop} : (\text{base} = \text{base})$$

Can we make sense of this?

The circle (first try)

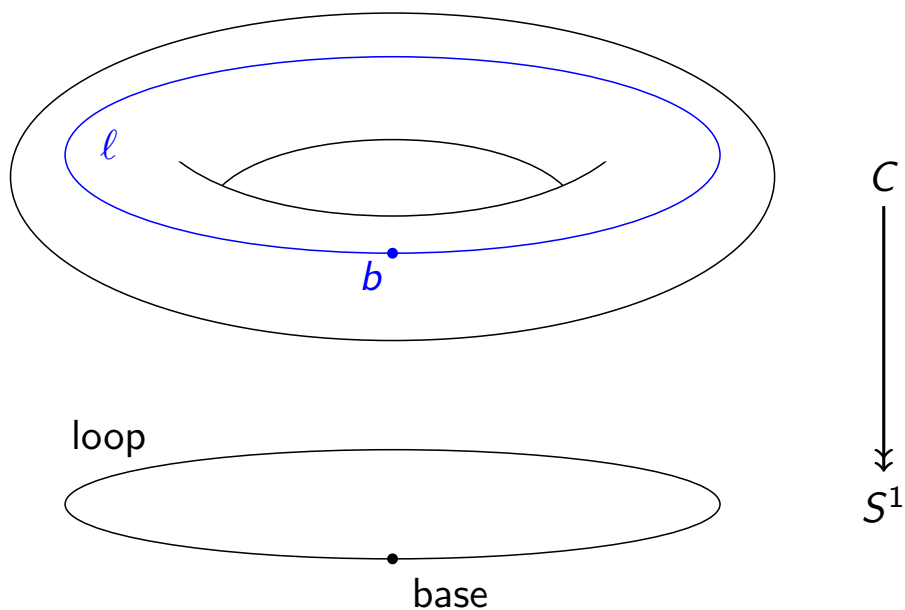
- ① **Formation**: There is a type S^1 .
- ② **Introduction**: $\text{base} : S^1$ and $\text{loop} : (\text{base} = \text{base})$.
- ③ **Elimination**: Given $b : C$ and $\ell : (b = b)$, for any $p : S^1$ we have $\text{match}(p, b, \ell) : C$.
- ④ **Computation**: $\text{match}(\text{base}, b, \ell)$ computes to b , and $\text{map}(\text{match}(-, b, \ell), \text{loop})$ computes to ℓ .

What about a dependent eliminator?

Dependent loops

As hypotheses of the **dependent** eliminator for S^1 , we need

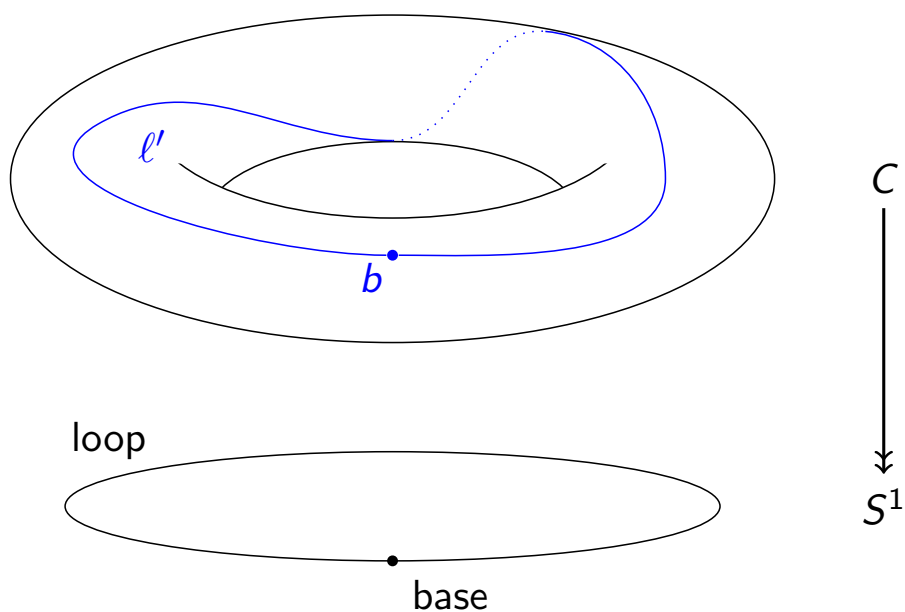
- 1 A point $b: C(\text{base})$.
- 2 A path ℓ from b to b lying **over** "loop".



Dependent loops

As hypotheses of the **dependent** eliminator for S^1 , we need

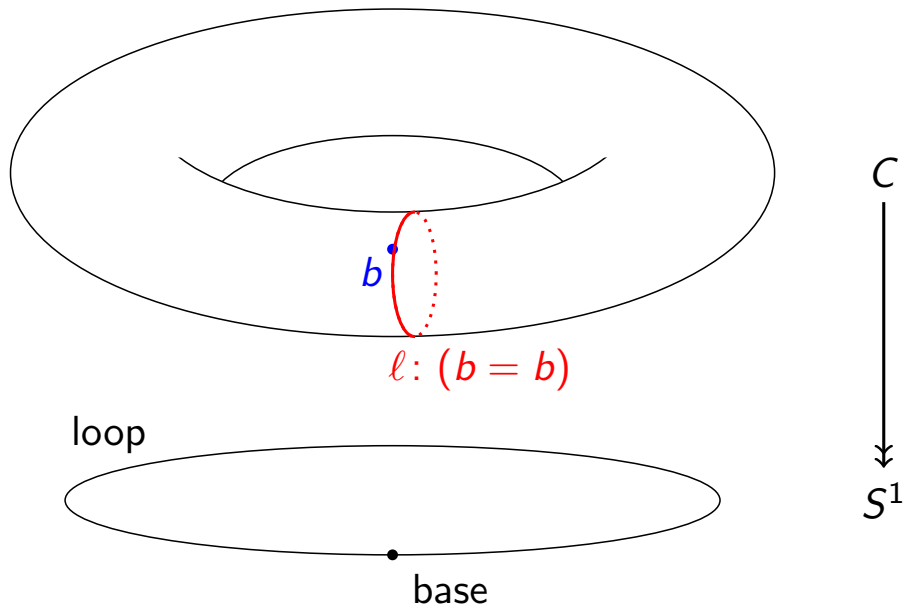
- 1 A point $b: C(\text{base})$.
- 2 A path ℓ' from b to b lying **over** "loop".



Dependent loops

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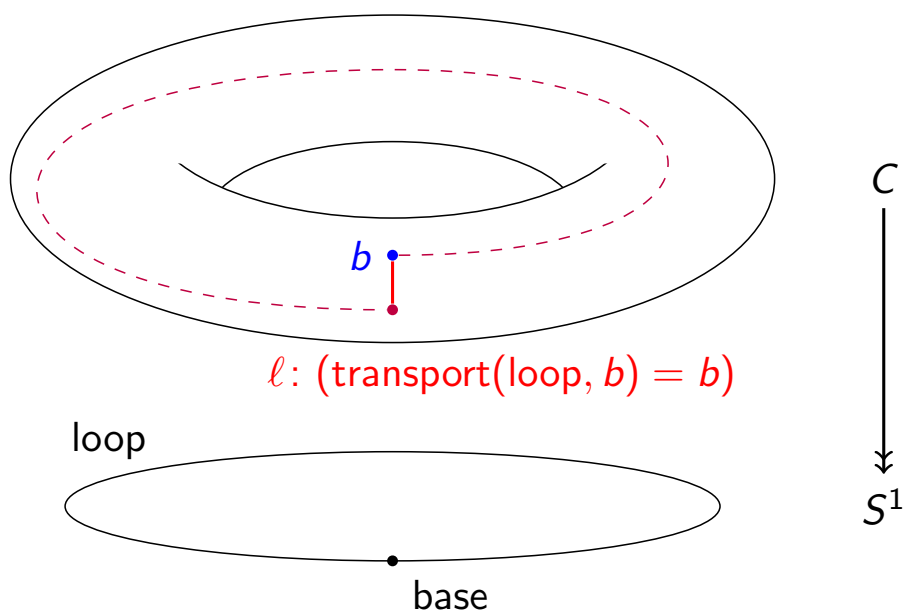
- 1 A point $b: C(\text{base})$.
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Dependent loops

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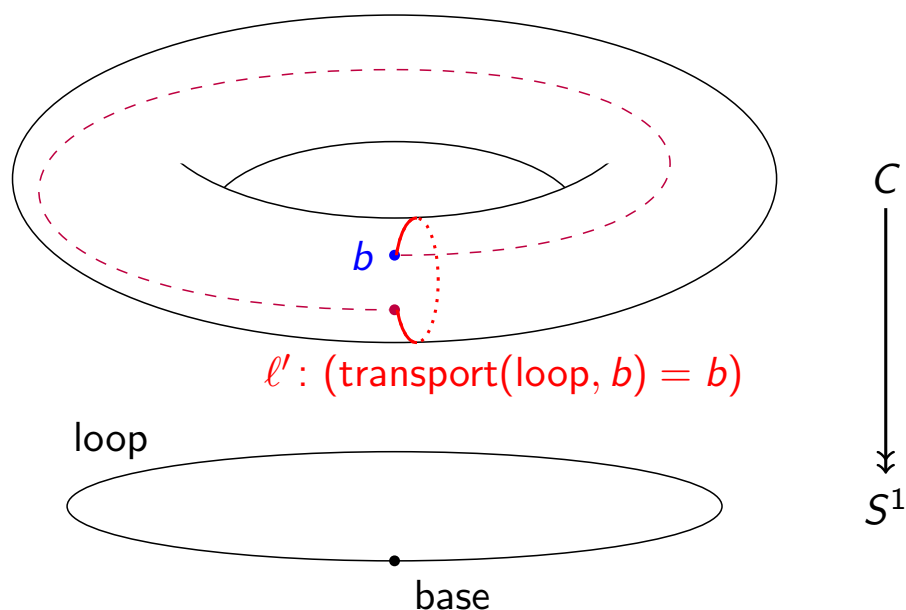
- 1 A point $b: C(\text{base})$.
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Dependent loops

As hypotheses of the **dependent** eliminator for S^1 , we need

- 1 A point $b: C(\text{base})$.
- 2 A path ℓ from b to b lying **over** “loop”.



The circle (final version)

- 1 **Formation:** There is a type S^1 .
- 2 **Introduction:** $\text{base} : S^1$ and $\text{loop} : (\text{base} = \text{base})$.
- 3 **Elimination:** Given $b: C(\text{base})$ and $\ell: (\text{trans}(\text{loop}, b) = b)$, for any $p: S^1$ we have $\text{match}(p, b, \ell) : C(p)$.
- 4 **Computation:** $\text{match}(\text{base}, b, \ell)$ computes to b , and $\text{map}(\text{match}(-, b, \ell), \text{loop})$ computes to ℓ .

The Interval

Example

The **interval** I is an inductive type with three constructors:

$$\text{zero} : I \quad \text{one} : I \quad \text{segment} : (\text{zero} = \text{one})$$

- Unsurprisingly, this type is provably contractible.
- But surprisingly, it is not useless; it implies function extensionality.

The 2-sphere

Example

The **2-sphere** S^2 has two constructors:

$$\text{base2} : S^2 \quad \text{loop2} : (\text{refl}_{\text{base2}} = \text{refl}_{\text{base2}})$$

OR:

$$\text{northpole} : S^2$$
$$\text{southpole} : S^2$$
$$\text{greenwich} : (\text{northpole} = \text{southpole})$$
$$\text{dateline} : (\text{northpole} = \text{southpole})$$
$$\text{east} : (\text{greenwich} = \text{dateline})$$
$$\text{west} : (\text{greenwich} = \text{dateline})$$

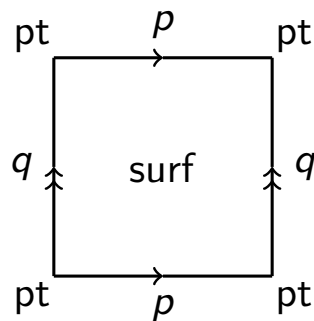
etc. . .

The torus

Example

The **torus** T^2 has four constructors:

$$\begin{aligned} \text{pt} &: T^2 \\ p &: (\text{pt} = \text{pt}) \\ q &: (\text{pt} = \text{pt}) \\ \text{surf} &: (p * q = q * p) \end{aligned}$$

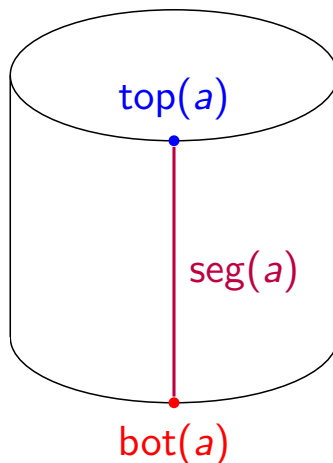


Cylinders

Example

The **cylinder** $\text{Cyl}(A)$ on A has three constructors:

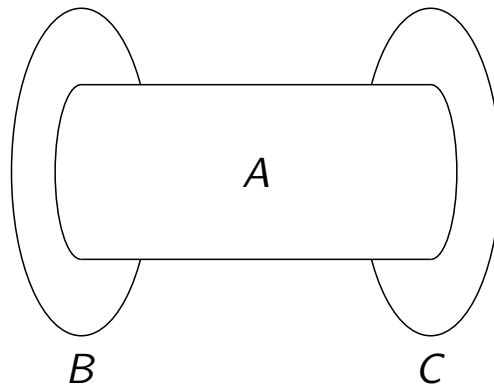
$$\begin{aligned} (a: A) \vdash (\text{top}(a) : \text{Cyl}(A)) & \quad (a: A) \vdash (\text{bot}(a) : \text{Cyl}(A)) \\ (a: A) \vdash (\text{seg}(a) : (\text{top}(a) = \text{bot}(a))) & \end{aligned}$$



Homotopy pushouts

Example

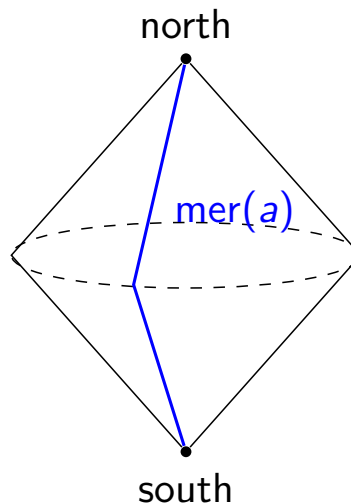
The **homotopy pushout** of $f: A \rightarrow B$ and $g: A \rightarrow C$ has three constructors:

$$(b: B) \vdash (\text{left}(b) : \text{pushout}(f, g))$$
$$(c: C) \vdash (\text{right}(c) : \text{pushout}(f, g))$$
$$(a: A) \vdash (\text{glue}(a) : (\text{left}(f(a)) = \text{right}(g(a))))$$


Suspension

Example

The **suspension** ΣA of A has three constructors:

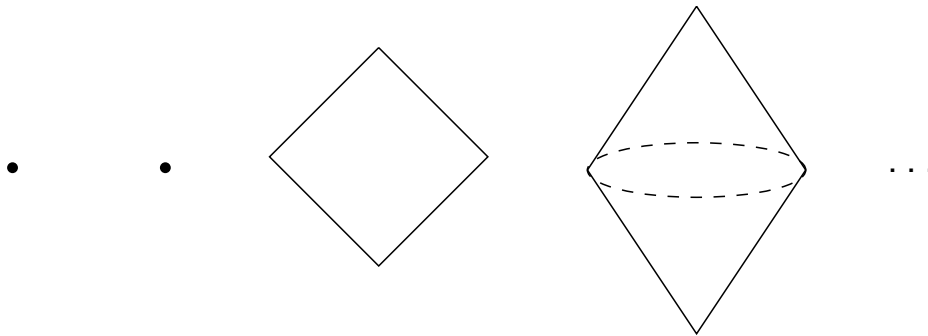
$$\text{north} : \Sigma A \qquad \text{south} : \Sigma A$$
$$(a: A) \vdash (\text{mer}(a) : (\text{north} = \text{south}))$$


Higher spheres

Example

The n -sphere S^n is defined by **recursion** on n :

$$S^0 := 1 + 1$$
$$S^{s(n)} := \Sigma(S^n)$$



Nontriviality

Theorem

The type S^1 is contractible \iff all types are h -sets.

Proof.

Easy; S^1 is the “universal loop”. □

HITs by themselves don't guarantee the homotopy theory is nontrivial. We need something else, like univalence.

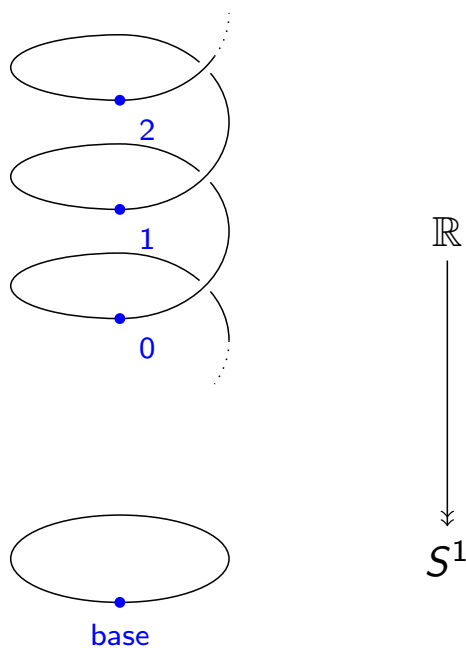
$\pi_1(S^1) \cong \mathbb{Z}$, classically

$$\pi_1(S^1) \cong \mathbb{Z}$$

How do we prove this classically?

- 1 Consider the winding map $\mathbb{R} \rightarrow S^1$.
- 2 This is the **universal cover** of S^1 .
- 3 Thus, its fiber over a point, namely \mathbb{Z} , is $\pi_1(S^1)$.

The universal cover of S^1



$\pi_1(S^1) \cong \mathbb{Z}$, homotopically

$$\pi_1(S^1) \cong \mathbb{Z}$$

A more homotopy-theoretic way to phrase the classical proof:

- ① We have a fibration $\mathbb{R} \rightarrow S^1$ with fiber \mathbb{Z} .
- ② We have a map $* \rightarrow S^1$, whose homotopy fiber is ΩS^1 .
- ③ \mathbb{R} is contractible, so we have an equivalence $* \simeq \mathbb{R}$ over S^1 .
By the short five lemma, the induced map on homotopy fibers is an equivalence.

$$\begin{array}{ccccc} \Omega S^1 & \longrightarrow & * & \longrightarrow & S^1 \\ \sim \downarrow & & \sim \downarrow & & \parallel \\ \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 \end{array}$$

- ④ In particular, $\pi_1(S^1) \cong \mathbb{Z}$.

$\pi_1(S^1) \cong \mathbb{Z}$, type-theoretically

How can we build the fibration $\mathbb{R} \twoheadrightarrow S^1$ in type theory?

- A fibration over S^1 is a dependent type $R: S^1 \rightarrow \text{Type}$.
- By the eliminator for S^1 , a function $R: S^1 \rightarrow \text{Type}$ is determined by
 - A point $B: \text{Type}$ and
 - A path $\ell: (B = B)$.
- By univalence, ℓ is an equivalence $B \simeq B$.

Thus we can take $B = \mathbb{Z}$ and ℓ to be “+1”.

- All that's left to do is prove that $\sum_{x: S^1} R(x)$ is contractible. We can do this by “induction” on S^1 .
- What we get is $\Omega S^1 \cong \mathbb{Z}$, which is classically stronger than $\pi_1(S^1) \cong \mathbb{Z}$. Here, we don't yet have a definition of π_1 .

Recall: A is (-1) -truncated, or an **h-prop**, if

$$\prod_{x,y:A} (x = y).$$

The **support** of A , denoted $\text{supp}(A)$, is supposed to be:

- an h-prop that contains a point precisely when A does.
- a reflection of A into h-props.

Support as an HIT

Definition (Lumsdaine)

The **support** of A is inductively defined by two constructors:

$$\begin{aligned} (a : A) &\vdash (\text{inhab}(a) : \text{supp}(A)) \\ (x : \text{supp}(A)), (y : \text{supp}(A)) &\vdash (\text{inpath}(x, y) : (x = y)) \end{aligned}$$

The type of `inpath` is precisely `isProp(supp(A))!`

- ③ if $(x : A) \vdash (c_A : C)$ and $(z, w : C) \vdash (c_{=} : (z = w))$, for any $p : \text{supp}(A)$ we have $\text{match}(p, c_A, c_{=}) : C$.

The hypotheses of the eliminator say exactly that C is an h-prop and we have a map $A \rightarrow C$.

$$\begin{array}{ccc} A & & \\ \text{inhab} \downarrow & \searrow^{c_A} & \\ \text{supp}(A) & \cdots \rightarrow & C \end{array}$$

The rest of logic

$$\begin{aligned} P \text{ and } Q &\iff P \times Q \\ P \text{ implies } Q &\iff Q^P \\ \top \text{ (true)} &\iff \mathbf{1} \\ \perp \text{ (false)} &\iff \emptyset \\ (\forall x: A)P(x) &\iff \prod_{x: A} B(x) \\ \\ P \text{ or } Q &\iff \text{supp}(P + Q) \\ (\exists x: A)P(x) &\iff \text{supp}(\sum_{x: A} B(x)) \end{aligned}$$

The magic of supp

Note: our ability to define “isProp” **without** using “supp” was crucial to our ability to **define** “supp” itself!

- Because we defined isProp using only paths, path-constructors can “universally force” a type to be an h-prop.
- Because isProp is an h-prop, these path-constructors have no other effect (give no extra data).

0-truncation

Example

The **0-truncation** $\pi_0(A)$ has two constructors:

$$(a : A) \vdash (\text{cpnt}(a) : \pi_0(A))$$
$$(x, y : \pi_0(A)), (p, q : (x = y)) \vdash (\text{pp}(x, y, p, q) : (p = q))$$

- The type of pp is precisely $\text{isHlevel}(2, A)$.
- The eliminator says that $\pi_0(A)$ is a reflection of A into h-sets.

Now we can define

$$\pi_1(A) := \pi_0(\Omega A)$$

etc....

Nonclassicality

Remark

h-sets and homotopy groups are a bit surprising.

- ① A map $f : A \rightarrow B$ which induces $\pi_n(A) \xrightarrow{\sim} \pi_n(B)$ for all $n : \mathbb{N}$ is **not** necessarily an equivalence!
 - Not closely related to non-CW-complex spaces.
 - It has to do with **non-hypercomplete** $(\infty, 1)$ -toposes.
 - A reason not to call “equivalences” “weak equivalences”.
- ② There may be types which do not admit a connected map from an h-set!
 - This happens in $\infty\text{Gpd}/X$ if X is not discrete.
 - As a foundation, not every ∞ -groupoid has an “underlying set” of objects (though it does have a π_0).
 - In particular, not every type has a cell decomposition.

These are “classicality properties” of ∞Gpd , like excluded middle and the axiom of choice in Set .

Localization

Given $f: A \rightarrow B$.

Definition

- Z is **f -local** if $Z^B \xrightarrow{-\circ f} Z^A$ is an equivalence.
- An **f -localization** of X is a reflection of X into f -local spaces.

Examples

- If f is $S^n \rightarrow D^{n+1}$, then f -local means $(n - 1)$ -truncated.
- Localization and completion at primes.
- Construction of $(\infty, 1)$ -toposes from $(\infty, 1)$ -presheaves.
- ...

h-isomorphisms

Recall: $f: A \rightarrow B$ is an **h-isomorphism** if we have

- A map $g: B \rightarrow A$
- A homotopy $r: \prod_{a: A} (g(f(a)) = a)$
- A map $h: B \rightarrow A$
- A homotopy $s: \prod_{b: B} (f(g(b)) = b)$

The type $\text{isHiso}(f)$ is an h-prop, equivalent to $\text{isEquiv}(f)$.

Definition

Given $f: A \rightarrow B$ and X , the **localization** $L_f X$ has constructors:

$$\begin{aligned} & (x: X) \vdash (\text{tolocal}(x) : L_f X) \\ & (g: A \rightarrow L_f X), (b: B) \vdash (\text{lsec}(g, b) : L_f X) \\ & (g: A \rightarrow L_f X), (a: A) \vdash (\text{lsech}(g, a) : (\text{lsec}(g, f(a)) = g(a))) \\ & (g: A \rightarrow L_f X), (b: B) \vdash (\text{lret}(g, b) : L_f X) \\ & (h: B \rightarrow L_f X), (b: B) \vdash (\text{lreth}(h, b) : (\text{lret}(h \circ f, b) = h(b))) \end{aligned}$$

The meaning of localization

- Of course, tolocal is a map $X \rightarrow L_f X$.
- lsec is a map $(L_f X)^A \rightarrow (L_f X)^B$.
- lsech is a homotopy from $(L_f X)^A \xrightarrow{\text{lsec}} (L_f X)^B \xrightarrow{-\circ f} (L_f X)^A$ to the identity.
- lret is a map $(L_f X)^A \rightarrow (L_f X)^B$.
- lreth is a homotopy from $(L_f X)^B \xrightarrow{-\circ f} (L_f X)^A \xrightarrow{\text{lret}} (L_f X)^B$ to the identity.

Together, $(\text{lsec}, \text{lsech}, \text{lret}, \text{lreth})$ exactly inhabit “ $\text{isHiso}(- \circ f)$ ”, i.e. “ $\text{isLocal}(f, X)$ ”.

Thus, $L_f X$ is an f -localization of X .

The other factorization

Recall:

- A model category has two weak factorization systems:
(acyclic cofibrations, fibrations)
(cofibrations, acyclic fibrations)
- Identity types correspond to the first WFS, using the **mapping path space**:

$$A \rightarrow [y: B, x: A, p: (g(x) = y)] \twoheadrightarrow B$$

- In topology, the second WFS is likewise related to the **mapping cylinder**.

$$A \rightarrow Mf \twoheadrightarrow B$$

Can we use HITs to construct this?

Acyclic fibrations

What is an **acyclic fibration** in type theory?

- ① A fibration that is also an equivalence.
- ② A fibration $p: B \twoheadrightarrow A$ which admits a section $s: A \rightarrow B$ (hence $ps = 1_A$) such that $sp \sim 1_B$.
- ③ A dependent type $B: A \rightarrow \text{Type}$ such that each $B(a)$ is contractible.

What is a **cofibration** in type theory?

Actually, what is an **acyclic cofibration** in type theory?

I.e. when does $i: A \rightarrow B$ satisfy $i \sqsupseteq p$ for any fibration p ?

Acyclic cofibrations

Theorem (Gambino-Garner)

If B is an inductive type and i is its **only** constructor, then $i \sqsupseteq p$ for any fibration p .

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \overset{?}{\nearrow} & \downarrow p \\ B & \xrightarrow{g} & X \end{array}$$

Proof.

- p is a dependent type $Y: X \rightarrow \text{Type}$; we want to define

$$h: \prod_{b: B} Y(g(b))$$

- By the eliminator, it suffices to specify $h(b)$ when $b = i(a)$.
- But then we can take $h(i(a)) := f(a)$. □

Path object factorizations

Example

$\text{refl}: A \rightarrow \text{Id}_A$ is the only constructor of the identity type. Thus,

$$A \xrightarrow{\text{refl}} \text{Id}_A \twoheadrightarrow A \times A$$

is an (acyclic cofibration, fibration) factorization.

Some cofibrations

Theorem

If B is an inductive type and $i: A \rightarrow B$ is **one** of its constructors, then $i \sqsubset p$ for any **acyclic** fibration p .

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \overset{?}{\dashrightarrow} & \downarrow p \\ B & \xrightarrow{g} & X \end{array}$$

Proof.

- Now we have a section $s: \prod_{x: X} Y(x)$.
- We define $h: \prod_{b: B} Y(g(b))$ with the eliminator of B :
 - If $b = i(a)$, take $h(b) := f(a)$.
 - If b is some other constructor, take $h(b) := s(g(b))$. □

More cofibrations

Theorem

If B is a **higher** inductive type and $i: A \rightarrow B$ is one of its point-constructors, then $i \sqsupseteq p$ for any acyclic fibration p .

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow i & \nearrow ? & \downarrow p \\ B & \xrightarrow{g} & X \end{array}$$

Proof.

- Now we have a section $s: \prod_{x: X} Y(x)$.
- We define $h: \prod_{b: B} Y(g(b))$ with the eliminator of B :
 - If $b = i(a)$, take $h(b) := f(a)$.
 - If b is some other point-constructor, take $h(b) := s(g(b))$.
 - **In the case of path-constructors, use the contractibility of the fibers of p .** \square

The other factorization

Need a mapping cylinder for $f: A \rightarrow B$ that is **dependent over B** .

Definition

The **mapping cylinder** $Mf: B \rightarrow \text{Type}$ has three constructors:

$$\begin{aligned} (b: B) &\vdash (\text{right}(b) : Mf(b)) \\ (a: A) &\vdash (\text{left}(a) : Mf(f(a))) \\ (a: A) &\vdash (\text{glue}(a) : (\text{left}(a) = \text{right}(f(a)))) \end{aligned}$$

Theorem (Lumsdaine)

- This defines a WFS (cofibrations, acyclic fibrations).
- With the other WFS, and the type-theoretic equivalences, we have a **model category** (except for strict limits and colimits).

Categorical models

Conversely:

Theorem (Lumsdaine–Shulman)

*A well-behaved combinatorial model category which models type theory as before (lccc etc.) also models all **higher inductive types**.*

(In particular, simplicial sets.)

Very rough sketch of proof.

Combine the transfinite construction of initial algebras with the homotopy-theoretic small object argument. □

Elementary $(\infty, 1)$ -toposes

Proposal

An **elementary $(\infty, 1)$ -topos** is an $(\infty, 1)$ -category \mathcal{C} such that:

- ① \mathcal{C} has finite limits.
- ② \mathcal{C} is locally cartesian closed.
- ③ \mathcal{C} has sufficiently many object classifiers.
- ④ \mathcal{C} has sufficiently many “higher initial algebras”
($\Rightarrow \mathcal{C}$ has finite colimits).

Conjecture

Any elementary $(\infty, 1)$ -topos has an internal homotopy type theory modeling the univalence axiom and higher inductive types.