Math 1260 - Final Exam

- You have 120 minutes to do this exam.
- No calculators!
- You may have one 3×5-card of notes
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

Problem	Total Points	Score
1	15	
2	15	
3	10	
4	15	
5	10	
6	20	
7	20	
EC	5	
Total	105	

1. True/False (1 point each)

(1) | **T** | **F** | The dot product $\langle 1,4,3\rangle \cdot \langle -1,1,2\rangle$ is equal to 9.

(2) | **T** | **F** | If a vector field $\mathbf{F}(x,y)$ is a gradient of some potential function $f(x,y)$ then the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ along any ellipse is zero.

(3) | **T** | **F** | The set $\{\phi = \pi/2, \rho > 0\}$ in spherical coordinates is the positive x-axis.

(4) | **T** | **F** | If $\mathbf{r}(t)$ is a curve in $\mathbb{R}^3$ then $\mathbf{r}'(t_0)$ gives the tangent vector to the curve at time t_0.

(5) | **T** | **F** | If $S = \{x^2+y^2 \le 5\}$, then in polar coordinates the corresponding region in the (r,θ)-plane is a rectangle.

(6) | **T** | **F** | It is possible that $\mathbf{v}\cdot\mathbf{w} > 0$ and $(\mathbf{v}+\mathbf{w})\cdot\mathbf{w} = 0$.

(7) | **T** | **F** | If the x-coordinate of the center of mass is zero (i.e. $\overline{x} = 0$) over a region $S \subset \mathbb{R}^2$, then $\iint_S x\,dA = 0$.

(8) | **T** | **F** | The vector field $\mathbf{F} = \langle y^2 - x, 2xy + y^2\rangle$ is a conservative vector field.

(9) | **T** | **F** | The point $(3,7,5)$ lies on both of the lines: $\mathbf{r}(t) = \langle t, t+4, 2t-1\rangle$ and $\mathbf{s}(t) = \langle 3t-3, 3t+1, 2t+1\rangle$.

(10) | **T** | **F** | Suppose $\mathbf{F}(x,y,z)$ is defined in terms of polynomial functions and has the property that curl$(\mathbf{F}) = \mathbf{0}$. Then $\mathbf{F}$ is conservative.

(11) | **T** | **F** | If S is a sphere with outward normal $\mathbf{n}$ and $\mathbf{F}$ has constant divergence then $\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = 0$.

(12) | **T** | **F** | In general, the work done by a conservative vector field is zero **only** if the path is closed.

(13) | **T** | **F** | If the divergence of $\mathbf{F}$ is constant, then $\nabla(\text{div } \mathbf{F}) = \mathbf{0}$.

(14) | **T** | **F** | A normal vector to the cylinder $x^2+z^2 = 4$ is always parallel to the y axis.

(15) | **T** | **F** | If f is a function with continuous second partial derivatives then $f_{yx} = -f_{xy}$.

2) **Short Answer** (5 points each)

A) Calculate

$$\int_0^5 \int_0^{\sqrt{25-y^2}} (x^2 + y^2)^{2015} \, dx \, dy.$$

B) Determine whether $\mathbf{F} = \langle y^2, 2xy, z^3 \rangle$ is a conservative vector field. If so, find its potential.

C) Let $z = x^2 \sin(xy) + y^2$. Compute $\frac{\partial z}{\partial x}$.

3. A) (5 Points) Find a parametrization of the line L through the center of the two spheres $x^2 + (y-1)^2 + z^2 = 1$, and $(x-5)^2 + y^2 + z^2 = 1$.

B) (5 points) Find the equation of the tangent plane to the surface $z = 5xy^2 - 3x^3 + 5y$ at the point where $x = 1, y = 2$.

4. A) (10 points) Johannes Kepler asked which cylinder of radius r and height $2h$ inscribed in the unit sphere has maximal volume. To solve this problem, use Lagrange multiplers to maximize the volume

$$V = 2\pi r^2 h$$

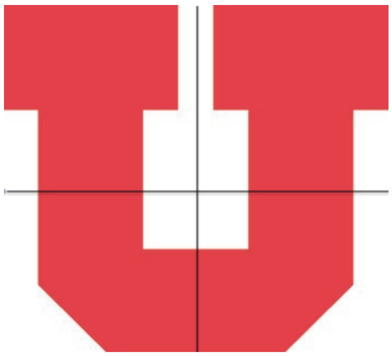
subject to the constraint $r^2 + h^2 = 1$. Write down the values of r, h, and V that achieve this maximum.

B) (5 points) Determine (with explanation) whether or not the limit $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-2y}$ exists. If so, compute it.

5) (10 points) If $\mathbf{F} = \langle xy, yz, z^2 \rangle$ and G is the rectangle with vertices $(0, 7, 2)$, $(0, 7, 3)$, $(4, 7, 2)$, $(4, 7, 3)$, and $\mathbf{n}$ is the normal that points in the positive y direction, compute $\iint_G \mathbf{F} \cdot \mathbf{n}\, dS$.

6) A) (10 points) Suppose that S is the red shape below. You may assume that it is symmetric about the y-axis, but not necessarily about the x-axis. Given the following data, determine the coordinates $(\overline{x}, \overline{y})$ of the center of mass of S:

$$Area(S) = 50, \quad \iint_S x + 2y + 3 \, dA = 25$$



B) (10 points) With S the same as above, suppose that C is the boundary curve of S oriented counterclockwise. Compute the following line integral:

$$\int_C \langle 5x - 5y, 75x^2 + y \rangle \cdot \, d\mathbf{r}.$$

7) A) (10 points) Evaluate $\iint_M \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{F} = \langle 3xy^2, 3x^2y, z^3 \rangle$ and M is the surface of the sphere of radius 6 centered at the origin. Assume $\mathbf{n}$ points outward.

B) (10 points) Let S be a surface in $\mathbb{R}^3$ that has a boundary given by $C = \{x^2 + y^2 = 4,\ z = 1\}$. That is, the boundary is a circle C in the plane $z = 1$. Let $\mathbf{F} = \langle \tan(e^{x^2}), x^2 z, y \rangle$. Given that

$$\iint_S \langle 1 - x^2, 0, 2xz \rangle \cdot \mathbf{n}\, dS = -3,$$

first, take a deep breath, and then compute

$$\left(\int_C \mathbf{F} \cdot d\mathbf{r} \right)^2.$$

Write a sentence or two explaining how you arrived at your answer. (This isn't meant to be so hard. Think about what Theorem you want to use and then see if the ingredients are there...)

Extra Credit. Let S be the surface defined by $z = e^{\sin(xy)} x^2 y^4 (1 - x - y)^5, z \geq 0, x \geq 0, y \geq 0$. Suppose that the volume of the region below S and above the $z = 0$ plane is C. If $\mathbf{F} = \langle y, z, \frac{1}{C}z + x \rangle$, compute

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

with outward pointing normal.