Math 1260 - Final Exam Practice Solutions

- You have 120 minutes to do this exam.
- No calculators!
- You may have one 3×5-card of notes
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

Problem	Total Points	Score
1	20	
2	12	
3	8	
4	10	
5	10	
6	10	
7	10	
Total	80	

1. True/False (1 point each) (THERE WILL ONLY BE 15 ON THE ACTUAL EXAM)

(1) | **T** | **F** | The dot product between $\langle 1,1,2\rangle$ and $\langle 2,3,4\rangle$ is 13. TRUE

(2) | **T** | **F** | If $\mathbf{r}(t)$ is a curve in $\mathbb{R}^3$ then then $\mathbf{r}'(t_0)$ gives the vector perpendicular to the curve at time t_0. FALSE (this is a tangent vector)

(3) | **T** | **F** | If $\nabla f(3,1) = \langle 0,0\rangle$ and $f_{xx}(3,1) > 0$ then $(3,1)$ is a local minimum of f. FALSE, you need information about the Hessian determinant.

(4) | **T** | **F** | The triple integral $\iiint_E \operatorname{div}\mathbf{F}\, dV$ is always zero since the flux of $\mathbf{F}$ through the boundary surface is zero. FALSE (we've seen lots of examples where we used the divergence theorem and gotten nonzero answers)

(5) | **T** | **F** | For any two unit vectors $\mathbf{v}, \mathbf{w}$ we have $\|\mathbf{v}\times\mathbf{w}\|^2 + (\mathbf{v}\cdot\mathbf{w})^2 = 1$. TRUE since the vectors have length one, we have $\|\mathbf{v}\times\mathbf{w}\|^2 + (\mathbf{v}\cdot\mathbf{w})^2 = \sin^2 + \cos^2$ of the angle between the vectors.

(6) | **T** | **F** | The length of the gradient $|\nabla f|$ is maximal at a maximum of f. FALSE At a maximum the gradient should be zero, which is the smallest possible value of the magnitude of a vector!

(7) | **T** | **F** | The curl of a conservative vector field in $\mathbb{R}^3$ is zero. TRUE

(8) | **T** | **F** | It is possible that $\mathbf{v}\cdot\mathbf{w} > 0$ and $\mathbf{v}\times\mathbf{w} = \mathbf{0}$. TRUE. The just take any nonzero $v = w$ for instance.

(9) | **T** | **F** | The lines parametrized by $\mathbf{r}(t) = \langle t+1, t, -t, \rangle$ and $\mathbf{s}(t) = \langle t, t, -t, \rangle$ do not intersect. TRUE. We could set this up by solving $(t+1, t, -t) = (s, s, -s)$. We'd see that $t = s$ and also $t + 1 = s$, which is impossible.

(10) | **T** | **F** | If $\mathbf{u}\cdot\mathbf{v} = 0$ then $\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = 0$. TRUE. Just look at the formula for the projection

(11) | **T** | **F** | The volume of a solid can be computed as the flux of the field $\langle 0, y, 0\rangle$ through the boundary surface. TRUE - the divergence of this field is 1, so the divergence theorem tells us that the flux through the boundary is equal to the volume.

(12) | **T** | **F** | The distance between the unit sphere centered at $(0,0,0)$ and the plane $z = 5$ is equal to 4. TRUE. The closest point to the plane is the point $(0,0,1)$ which is a distance of 4 away

(13) | **T** | **F** | The line $\mathbf{r}(t) = \langle t, t, t\rangle$ is perpendicular to the plane $x + y + z = 10$. TRUE. The normal vector to the plane is $\langle 1,1,1\rangle$ which is the direction of the line.

(14) | **T** | **F** | If the divergence of a vector field $\mathbf{F}$ is always positive inside of a solid E then $\iint_{\partial E} \mathbf{F}\cdot\mathbf{n}\, dS > 0$ if the normal is taken to point outward. TRUE by the divergence theorem

(15) | **T** | **F** | If E is a solid with unit volume, then $\iiint_E x\, dV$ is equal to the x coordinate of the center of mass of E. TRUE

(16) | **T** | **F** | If S is a sphere with outward normal $\mathbf{n}$ and $\mathbf{F}$ is a constant vector field then $\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = 0$. TRUE again by the divergence theorem

(17) | **T** | **F** | Green's Theorem holds for a region with a hole provided the complete boundary of the region is oriented correctly. TRUE

(18) | **T** | **F** | The work done by a conservative vector field only depends on the endpoints of the path. TRUE

(19) | **T** | **F** | If the divergence of $\mathbf{F}$ is zero everywhere in a solid E then there is some point in E where curl $\mathbf{F} = \mathbf{0}$. FALSE. Consider the vector field $(y, 0, 0)$

(20) | **T** | **F** | Multiplying each component of a vector $\mathbf{v}$ by the scalar a multiples the length of $\mathbf{v}$ by a. FALSE it multiplies by $|a|$.

2. Short Answer (3 points each)

A) Write the equation of a circle of radius 2 centered at the origin in **polar coordinates**.

Solution: $r = 2$.

B) What is the unit normal vector to the cylinder $x^2 + y^2 = 8$ at the point $(2, 2, 5)$? (Hint: Draw a picture)

Solution: The z component will be zero, and the (x, y) components will point away from the origin, so the direction is $(2, 2, 0)$ and the normal vector is $(2, 2, 0)/\sqrt{8}$. I didn't specify inward or outward, so $\pm$ this would be correct.

C) Find an equation for the tangent plane to the surface $z = x^3 y - y$ at the point $(1, 2, 0)$.

Solution: Our formula for tangent planes was $z = z_0 + \nabla(f(x_0, y_0)) \cdot (x - x_0, y - y_0)$. So we have $\nabla f = (3x^2 y, x^3 - 1)$, which is $(6, 0)$ at $(1, 2)$, so the plane is given by

$$z = 0 + (6, 0) \cdot (x - 1, y - 2)$$
$$z = 6(x - 1).$$

D) Find a nonzero vector perpendicular to the plane $z = \frac{1}{2}x - 3y + 7$.

Solution: The normal vector is $(1/2, -3, -1)$, so any nonzero multiple of this will work.

3. A) (5 Points) Find the points on the surface $2z^2 = 3xy + 8$ that are closest / further from the origin.

Solution: This is a Lagrange multipliers problem. We want to optimize the function $f = x^2+y^2+z^2$ subject to the constraint that $2z^2 = 3xy+8$. Let's have our constraint be $g(x,y,z) = 2z^2 - 3xy - 8$. Before we dive in, notice that there are going to be points of the form (a million, a million, really big) so we don't expect to find a maximum distance. We set it up:

$$\nabla f = \lambda \nabla g$$

$$(2x, 2y, 2z) = \lambda(-3y, -3x, 4z)$$

This tells us $x = -3y\lambda/2$; $y = -3x\lambda/2$ and $z = 2\lambda z$. Let's sub the first equation into the second: We get $y = -3(-3y\lambda/2)\lambda/2 = 9y\lambda^2/4$. If $y = 0$ then we'd have also $x = 0$, and then using the constraint, we see that $z = \pm\sqrt{4} = \pm 2$. So potential critical points are $(0, 0, \pm 2)$.

Now let's assume $y \neq 0$. Then $\lambda = \pm 2/3$. This tells us that $x = \pm y$ and $z = 0$ (plug this value of λ back into the equations.) Thus our constraint tells us that

$$3xy + 8 = 2z^2$$

$$3x(\pm x) + 8 = 0$$

so $x = \pm\sqrt{8/3}$. Notice that only one value of λ, namely $\lambda = 2/3$ gave us solutions. The other gave imaginary numbers. Thus we see that our critical points are

$$(\sqrt{8/3}, -\sqrt{8/3}, 0), \ (-\sqrt{8/3}, \sqrt{8/3}, 0).$$

Let's check the distances of these points. $(0, 0, \pm 2)$ is distance 2. The other points are of distance $\sqrt{8/3 + 8/3 + 0} = \sqrt{16/3}$. This is bigger than 2. Thus the points that are closest to the origin are $(0, 0, 2)$ and $(0, 0, -2)$.

B) (3 points) Find a parametrization $\mathbf{r}(t) = P + t\mathbf{v}$ of the line defined by $x = 1, y = 1$.

Solution: We can just take the point $(1, 1, 0)$ as P and the direction is the direction of the z-axis, so we get $\mathbf{r}(t) = (1, 1, t)$.

4. A) (5 points) Let $f(x, y, z) = x + y + z$. Compute the directional derivative of f in the direction of the vector $(3, 4, 5)$

Solution: The unit vector in this direction is $(3, 4, 5)/\sqrt{50}$ so the answer is $\nabla(f)\cdot(3, 4, 5)/\sqrt{50} = (1, 1, 1)\cdot(3, 4, 5)/\sqrt{50} = 12/\sqrt{50}$. Notice that this is independent of the point (x, y, z) where we take the derivative. This makes sense because every point on the plane looks the same.

B) (5 points) If S is a solid sphere of radius a centered at the origin then compute the flux $\int_{\partial S} \mathbf{F}\cdot\mathbf{n}\, dS$ where $\mathbf{F} = \langle 3x, 3y + \sin(xz), z^2\rangle$ where $\mathbf{n}$ is the outward normal.

Solution: This is a divergence theorem question. The divergence of $\mathbf{F}$ is $3 + 3 + 2z = 6 + 2z$. Thus we must compute

$$\iiint_S 6 + 2z dV = 6\iiint dV + 2\iiint z dV.$$

The second integral is zero by symmetry. The first is just 6 times the volume of the sphere, so the answer is $8\pi a^3$.

5. A) (5 points) Suppose a roller coaster is in the shape of the curve

$$\mathbf{r}(t) = \langle 4\cos(t) + \sin(5t), 4\sin(t) - 2\cos(5t), 4\sin(2t)\rangle$$

and assume that a force $\mathbf{F} = \langle x^{100}, y, z^{20}\rangle$ acts on the car. What work is done by the force when going from $\mathbf{r}(0)$ to $\mathbf{r}(\pi)$?

Solution: $\mathbf{F}$ is conservative, and $\mathbf{F} = \nabla(x^{101}/101 + y^2/2 + z^{21}/2)$. Thus the work done is $f(end) - f(start)$ which is $f(\mathbf{r}(\pi)) - f(\mathbf{r}(0))$.

$$\mathbf{r}(0) = (4, -2, 0),\ \ \mathbf{r}(\pi) = (-4, 2, 0)$$

so the answer is

$$((-4)^{101}/101 + 2^2/2) - (4^{101}/101 + (-2)^2/2)$$
$$= -2 \cdot 4^{101}$$

B) (5 points) Find the volume of the solid inside the cylinder

$$x^2 + y^2 \leq 2$$

sandwiched between the graphs of $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 + 4$.

Solution: This is a standard double integral.

$$\iint_S (x^2 + y^2 + 4) - (x - y)\, dA$$

We should use polar coordinates. We get

$$\int_0^{2\pi} \int_0^{\sqrt{2}} (r^2 + 4 - r\cos\theta + r\sin\theta) r\, dr\, d\theta.$$
$$= 10\pi.$$

6. A) (5 points) Suppose that C is an ellipse centered around the point $(4, 3)$ in the xy plane. Suppose that C bounds a region of area 4π. If $\mathbf{F} = \langle 4x^2 + y^2, xy + x\rangle$ then compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is oriented counterclockwise.

Solution: This sounds like a Green's Theorem problem. We need to compute

$$\iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \iint_S y + 1 - 2y\, dA = \iint_S 1 - y\, dA$$

where S is the inside of the ellipse. This integral should be

$$Area(S) - \overline{y} \cdot Area(S) = 4\pi - 3(4\pi) = -8\pi.$$

B) (5 points) If $\mathbf{F} = \langle x, y^2, yz\rangle$ and S is the surface given by $y = 4$ with shadow a circle of radius 3 in the xz plane, compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is pointing away from the xz plane.

Solution: The surfcae is just a circle floating in the plane $y = 4$. The normal vector is $(0, 1, 0)$. Thus $\mathbf{F} \cdot n = y^2 = 16$. Thus our integral is

$$\iint_S 16\, dS$$

which is just 16 times the surface area, so the answer is

$$16 \cdot (\pi \cdot 3^2).$$

7. (Hardest) Let **F** be the vector field $\langle y^3, x - x^3 \rangle$.

a) (5 points) Find the closed curve C in the plane such that the integral $\mathbf{F} \cdot d\mathbf{r}$ has the largest possible value. (Hint: Think about what theorem might help you!)

b) (5 points) Compute this largest value.

Solution: This one is another Green's Theorem problem, so good for you if you thought to use it! We know that if the curve C is oriented counterclockwise then the integral will be equal to

$$\iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \iint_S 1 - 3x^2 - 3y^2 \, dA$$

where S is the interior of the curve. Naturally if we want this to be biggest, we should make sure that the curve encloses no points where this function is negative. So we want

$$1 - 3x^2 - 3y^2 > 0$$

in other words we want $x^2 + y^2 < 1/3$, so we are talking about a circle of radius $1/\sqrt{3}$. The boundary encloses precisely all the points where this function is positive, and if we add all those up , we will get the largest possible value.

That value is equal to

$$\int_0^{2\pi} \int_0^{1/\sqrt{3}} (1 - 3r^2) r \, dr d\theta$$
$$= 2\pi(1/2 \cdot 1/3 - 3/4 \cdot 1/9) = \pi/6.$$