WORKSHEET 1 - ON DEFINITIONS AND VECTORS
DUE WEDNESDAY SEPT. 2, 2015

You are greatly encouraged to work on this worksheet in groups - in fact you can turn in one worksheet per group, provided it is written neatly. We'll have some time to work on the worksheet in class but you are encouraged to continue working outside of class as well. You can write directly on the worksheet, or on a separate sheet of paper.

This worksheet has two important goals:

(1) Learning to read and understand a mathematical definition. In going through this worksheet, you should expect things to be a bit confusing at first. This is normal. Reading and understanding definitions is very important but also a slow process.

(2) Learning some properties of vectors as they relate to linear spaces. (Baby linear algebra).

We begin with a definition:

Definition: Let $\mathbf{v_1}, \ldots, \mathbf{v_k}$ be k vectors in $\mathbb{R}^n$. We define the **span** of $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ to be

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k\}, \text{where the } a_i \text{ are real numbers.}$$

In other words, the span is the set of all vectors that can be obtained by taking combinations of the v's and their multiples.

1: The first step in understanding a definition is to see what it means in a special case. Let's examine the case when $k = 1$ and $n = 2$. This is when we take 1 vector in $\mathbb{R}^2$. We want to understand

$$\operatorname{span}(\mathbf{v}_1) = \{a_1\mathbf{v_1}, \quad a_1 \in \mathbb{R}\}.$$

a) Draw a picture of $\operatorname{span}((1, 2))$, the span of the vector $(1, 2)$:

b) Draw a picture of $\operatorname{span}((0, 0))$, the span of the vector $(0, 0)$:

c) If $n = 3$ We are talking about vectors in $\mathbb{R}^3$. Draw a picture of span$((1,2,3))$, the span of the vector $(1,2,3)$ in $\mathbb{R}^3$:

d) Based off the above, when is span$(\mathbf{v})$ a line? (I'm asking for some condition on $\mathbf{v}$.)

2: Now let's consider the span of two vectors in $\mathbb{R}^2$.

a) What are k and n in this case?

b) What is span$((1,0),(0,1))$?

c) What is span$((1,6),(-2,-12))$?

You should have gotten two different geometric objects for b) and c).

d) If $\mathbf{v_1}, \mathbf{v_2}$ are two vectors in $\mathbb{R}^2$ then under what conditions will $\text{span}(\mathbf{v_1}, \mathbf{v_2})$ be a line, and when will it be a plane? (In this case that plane is the whole plane $\mathbb{R}^2$.)

3) Finally, let's consider the case when $k = 2$ and $n = 3$, that is when we have two vectors in $\mathbb{R}^3$.

Following up on previous parts, we can make a definition:

Definition: A **two dimensional plane** L **through the origin** is the set $\text{span}(\mathbf{v_1}, \mathbf{v_2})$ where v_1 are v_2 are two vectors satisfying your condition from 2.d.

Remember that earlier in class we said that planes where given by equations of the form $Ax+By+Cz = D$. If the plane contains in the origin then $D = 0$. (why?) Later in class we'll see how to define planes that don't pass through the origin.

a) If $L = \text{span}((1, 1, 0), (0, 0, 1))$ in $\mathbb{R}^3$, find an equation $Ax + By + Cz = D$ that defines this plane.

b) Conversely, find two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ such that the plane L defined by $2x + 3y - 5z = 0$ is given by

$$L = \text{span}(\mathbf{v_1}, \mathbf{v_2}).$$

4) (Back to $\mathbb{R}^2$:) Prove that the vectors (a, b) and (c, d) span all of $\mathbb{R}^2$ if and only if $ad - bc \neq 0$. (Hint: In 2.d you should have found a condition for the vectors to span a plane. Now just show two things: 1) If that condition is satisfied then $ad - bc \neq 0$. And 2) If $ad - bc \neq 0$ is satisfied then your condition from 2.d holds as well.

1. A bit of theory for the interested reader

If 2 vectors span a plane, we say that they are **linearly independent**. Similarly, if three vectors span a 3-dimensional space, we say they are linearly independent, etc etc. In a linear algebra course you'll learn about independence of vectors, spans of vectors, and more. Also, you might have noticed the formula for the determinant showing up in Problem 4. This will come up again when we talk about cross-products.