Worksheet 3 - Derivatives and Tangent Vectors
Due Wednesday Sept. 16, 2015

You are greatly encouraged to work on this worksheet in groups - in fact, write down the names of your group members' names, and contact information, so you can get in touch with them after class to finish up this worksheet:

When working on this worksheet go slowly - make sure every member of your group understands what is going on - it's not a race!

This week, we start to properly learn the calculus of *functions of more than one variable.* In previous calculus courses you have (mostly) studied functions of the form $y = f(x)$. In other words they had a single input "x" and a single output "y". Since the input and output are both in $\mathbb{R}$ we often write $f : \mathbb{R} \to \mathbb{R}$. Let's learn what this notation means:

$$f : X \to Y$$

means three things:

(1) f is the name of a function.
(2) f takes inputs in X - called the domain of f.
(3) f has values in Y - called the co-domain of f.

1) Match the following functions with the correct notation: (Hint: Think about what the INPUT and OUTPUT of each function is)

$$f(x, y) = x^2 + y^2 \qquad f(x) = \sin x + \cos x \qquad f(x) = \langle x, x^2 \rangle \qquad f(s, t) = \langle s^2 + t^2, 2st \rangle$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad f : \mathbb{R}^1 \to \mathbb{R}^1 \qquad f : \mathbb{R}^1 \to \mathbb{R}^2 \qquad f : \mathbb{R}^2 \to \mathbb{R}^1$$

Write down your own formulas for examples of $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^1 \to \mathbb{R}^3$.

2) Sometimes we might even have functions defined on other sets. For instance, the function

$$h : \{\text{Students in Math 1260}\} \longrightarrow \mathbb{Z}$$

$$h(A) = \text{age in years of } A$$

is a perfectly fine function. Similarly you could have a silly function like

$$g : \{\text{US States}\} \longrightarrow \{\text{Colors}\}$$

$$g(X) = \text{the favorite color of the state's oldest resident.}$$

Make up your own function between some interesting sets and write it down.

Sometimes we will define a function not on all of $\mathbb{R}$ (or $\mathbb{R}^2$, etc) but instead just on a subset. For instance, maybe in the function $f(t) = \langle \cos t, \sin t \rangle$ we only allow t to be in the range $[0, 2\pi]$. We could write that as $f : [0, 2\pi] \to \mathbb{R}$.

As another example, we might say something like: Let I be the interval $(0, \infty)$. We define $f : I \to \mathbb{R}$ as the function $f(x) = \log x$.

Remember: this is just notation and is meant to be helpful

In class last week we learned about *vector-valued functions*. These are functions whose values are vectors (as opposed to scalars). For example, the function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\mathbf{f}(t) = \langle t, t^2, t^3 \rangle$$

is a curve that we can think of as parameterizing the motion of a bug whose position at time t is $\mathbf{f}(t)$. In general, a function $\mathbf{f} : \mathbb{R} \longrightarrow \mathbb{R}^3$ is of the form

$$\mathbf{f}(t) = \langle x(t), y(t), z(t) \rangle$$

where we call the functions $x(t), y(t), z(t)$ the coordinate functions $\mathbf{f}$

3) If you had to guess - what to you think the *derivative* of $\mathbf{f}(t) = \langle t, t^2, t^3 \rangle$ would be? What about the limit as $t \to 0$ of $\mathbf{f}(t)$?

Let I be any interval (open or closed or half-open) of $\mathbb{R}$ and $\mathbf{f} : I \to \mathbb{R}^3$ we say that f is differentiable if for every point t in I, the limit

$$\lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$$

exists. Whatever this limit is, we denote it by $\mathbf{f}'(t)$, the derivative of $\mathbf{f}$.

4) Now write down the expression $\frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$ when $\mathbf{f}(t) = \langle x(t), y(t), z(t) \rangle$.

5) Now discuss with your group and write a sentence or two explaining why this shows that

$$\mathbf{f}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

If $\mathbf{f}(t)$ is a vector-valued function, we think of it describing the position of a particle (or bug). Then $\mathbf{f}'(t)$ is called the *velocity vector.* It's magnitude is the *speed* the particle is moving at time t, and its direction is... the direction the particle is moving. Sometimes we use the letter $\mathbf{r}$ to denote the position function. So we might write $\mathbf{v}(t) = \mathbf{r}'(t)$ which says that velocity is the derivative of the position function.

6) Find the speed and direction the particle is moving when $t = \pi/2$ if $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$

7) If t_0 is a time, then the *tangent vector of* $\mathbf{r}(t)$ *at time* t_0 is defined to be: the vector $\mathbf{r}'(t_0)$. Note - this vector is just a direction and a magnitude, so you can think of it as starting at the origin and ending wherever, but it's more helpful to think of it starting at the point $r(t_0)$ on the curve! Draw a picture of the curve from the previous problem and sketch the tangent vector.

8) However, if we talk about the tangent line to a curve, we really do want the line to go through the point. If $\mathbf{r}(t) = \langle \sin t, t^2, 1 - \cos t \rangle$ then what is the (parametric) equation of the tangent line at the point when $t = \pi/3$? (Hint: What was the equation for a line through a point P with direction $\mathbf{v}$? No need to simplify or combine the vectors.)

9) Suppose that $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a parameterized curve. We define the function $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$. This is called the unit tangent vector. Why do you think it is called this?

10) We will now show that $\mathbf{T}'(t)$ is always perpendicular to $\mathbf{T}(t)$. This means that no matter what time t you choose, if you find the vectors $\mathbf{T}(t)$ and $\mathbf{T}'(t)$, they will be perpendicular. First, explain why $\|\mathbf{T}(t)\| = 1$ implies that $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$.

11) Now use the property (to be discussed in class on Wednesday) that the derivative of $\mathbf{f}(t) \cdot \mathbf{g}(t)$ is equal to $\mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)$ to show that $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$, and hence that the functions are perpendicular.