Worksheet 5 - Open Sets
Due Wednesday Sept. 30, 2015

You are greatly encouraged to work on this worksheet in groups - in fact, write down the names of your group members' names, and contact information, so you can get in touch with them after class to finish up this worksheet:

When working on this worksheet go slowly - make sure every member of your group understands what is going on - it's not a race!

1) Open Sets

Yesterday in class, we learned what an open set is. Today we will write some proofs about open sets.

A) (1 point) Write down the definition of an open set:

B) (5 points) Prove that the first quadrant, i.e. the set of points

$$\{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$$

is open. Your proof should be well-written and should identify what the radius of the open ball at a point (a, b) should be.

C) (2 points) The empty set (the set with no points in it) **IS** an open set! Talk with your group about why this is true and write down your thoughts.

D) (7 Points) Show that if S and T are two open sets then their intersection, denoted $S \cap T$ is another open set. Hint: I recommend talking through this with you group and coming up with an argument using words, and then try to write down the proof. (Remember, to show that a set is open, we need to explain why for every point in that set, there's some open ball such that .... and that ball's radius is... etc)

E) (Extra Credit) By induction this problem shows that if $S_1, S_2, \ldots, S_n$ are a finite number open sets in $\mathbb{R}^2$ then so is the intersection of all of them $S_1 \cap \cdots \cap S_n$. Is this true if we allow an infinite number of open sets? Specifically, if $S_1, \ldots, S_n, \ldots$ is an infinite sequence of open sets then is the intersection of all of them open? Either prove or find a counterexample. Remember, the empty set **IS** open, so you won't find a counterexample there...

2) A function whose mixed partial derivatives are not equal

(10 Points) Let $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$ if $(x, y) \neq 0$ and $f(0, 0) = 0$. Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ by completing the following steps:

a) Compute a formula for $f_x(0, y)$ by using the definition of the partial derivative. Remember that

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

b) Similarly find a formula for $f_y(x, 0)$

c) Now compute $f_{yx}(0, 0)$. This is the partial derivative of f_y with respect to x at $(0, 0)$. Write this carefully as a limit and you should be able to compute it.

d) Similarly, compute $f_{xy}(0, 0)$. You should have a different answer than part c.

3) Some Odds and Ends

A) (5 points) Let

$$f(x,y) = \begin{cases} \frac{x^2-4y^2}{x-2y}, & \text{if } x \neq 2y \\ g(x), & \text{if } x = 2y. \end{cases}$$

If f is continuous on the whole plane, find a formula for $g(x)$.

B) (8 points) Below are collections of inequalities that describe a set of points in $\mathbb{R}^2$. Draw a sketch of each set and say whether or not the set is open. Which are open sets (no proofs needed)?

1) $x^2 + y^2 < 1$
2) $y > x^2$ and $|x| \leq 2$
3) $xy < 1$
4) $(x^2 + y^2 - 1)(4 - x^2 - y^2) > 0$. (Hint think about the sign of each factor)