Worksheet 6 - Lagrange Multipliers
Due Wednesday October 7th, 2015

You are greatly encouraged to work on this worksheet in groups - in fact, write down the names of your group members with contact information, so you can get in touch with them after class to finish up this worksheet:

When working on this worksheet go slowly - make sure every member of your group understands what is going on - it's not a race!

Our goal is to maximize a function $w = f(x, y)$ given a constraint $g(x, y) = k$ (an equation of a curve C).

1. (15 points total) Pictures will be really helpful for this worksheet, so we'll begin with one where we can draw everything. We'll be drawing graphs, and since this is an applied problem let's only draw them in the range $x, y \geq 0$.

In this problem, we want to compute the sizes of a rectangle that has the greatest area given that the length of its diagonal is 2. Say the rectangle has length x and width y.

The **constraint** is the condition that the length of the diagonal is 2. Write this as an equation $g(x, y) = k$. (1 point) Now draw a graph of points x, y that satisfy this equation. (1 point) We call this the **constraint curve**. Try to draw it in a different color than normal.

Since we want to optimize area, the function we want to optimize is given by $f(x, y) = xy$. For any pair of numbers, x, y this function returns the area. In your normal color, draw the level curves very carefully in your drawing for the level sets where $f(x, y) = c$ for $c = 0, 1, 2, 3, 4$. (2 points)

By looking at your picture, at what point on the constraint curve is the area maximized? (1 point) What is the relationship between the constraint curve and the level curve at that point? (2 points) What conclusion can you make about the gradient vectors to the curves $g(x, y)$ and $f(x, y)$ at the point? (2 points) Hint: Remember that the gradient of a function is perpendicular to its level sets.

The Theorem of Lagrange Multipliers says: To maximize or minimize a function $f(x, y)$ subject to the constraint $g(x, y) = k$, solve the system of equations

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$
$$g(x, y) = k$$

for (x, y) and λ. The pairs (x, y) are solutions to the extremum problem and λ is called a Lagrange multiplier.

(1 point) The first equation says that the gradient vectors of f and g are ______________

(1 point) The second equation says that (x, y) lies on the ______________

Now go back and use this method to solve the previous problem, to find x, y, λ. You should get the same answer as in the picture. (4 points)

2. (5 points) Suppose you need to make a rectangle out of 12 inches of wire. There sides are of length x and y. Suppose that one of the x sides also must have double thickness (it uses twice as much wire as its length). Find the rectangle of maximal area you can make with your wire.

Given a higher dimensional example, one can use the same method. Indeed, to find the maximum value of $w = f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, one can find the points where $\nabla f = \lambda \cdot \nabla g$ for some $\lambda \in \mathbb{R}$ and which lie on $g(x, y, z) = k$. The maximum and minimum are among these points.

3. (5 points) Use this method to find the points on the sphere $x^2 + y^2 + z^2 = 1$ closest and farthest from $Q = (1, 2, 3)$.

4. (5 points) Let $\mathbf{u} = (3\mathbf{i} - 4\mathbf{j})/5$ and $\mathbf{v} = (4\mathbf{i} + 3\mathbf{j})/5$ and suppose that at some point P we have $D_{\mathbf{u}} f = -6$ and $D_{\mathbf{v}} f = 17$.

a) Find ∇f at P

b) Check that $||\nabla f||^2 = D_{\mathbf{u}}^2 + D_{\mathbf{v}}^2$ in part (a). Show that this relationship is always true when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

5. (5 points) A tree trunk is shaped like a right circular cylinder. If the radius of the trunk is growing at the rate of 1/2 an inch per year and the height is increasing at a rate of 8 inches per year, then how fast is the volume increasing when the radius is 20 inches and the height is 400 inches?