THE COMPLEX NUMBERS

Some problems from today

(1) Find (and draw in the complex plane) the following when $z = 1 + 2i$ and $w = 3 - 4i$.

$$\overline{z}, \quad z + w, \quad 1/z, \quad zw, \quad w^2, \quad |z|$$

(2) Multiply the following two matrices together and see if you notice anythign?

$$Z = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

(3) Write down the formula for the product of $(a+bi)(c+di)$ in terms of a, b, c, d. Now compute the product of

$$Z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad W = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

(4) What is the formula for $(a+bi)+(c+di)$? What happens when you add the corresponding matrices?

(5) What is the length (absolute value) of $(a+bi)$? Does this quantity mean anything in matrix-land?

(6) What is the formula for the inverse of $(a+bi)$. What matrix would this correspond to?

(7) When you get to this point - let me know and i'll hand you the next part of the worksheet!

(8) You should be convinced that the operations of $+$ and $\cdot$ for complex numbers is the same as that for two by two matrices of the from $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

(9) For the moment, let's assume that $z = a + bi$ has length 1. So we have that $a^2 + b^2 = 1$. Talk about why this means that $a = \cos\theta$ for some angle t. Now explain why that same angle t has the property that $b = \sin\theta$.

(10) So you now know that your matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some angle θ. In other words - our complex number $a + bi$ corresponds to that matrix with trig functions! Now for the punchline, see what this matrix does to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Draw a picture of the where each vector starts and finishes. How could you describe what is happening?

(11) What you should have seen is that "multiplying by $a + bi$" is the same as rotating by θ. Suppose that we have another numbers $c + di$ with $c^2 + d^2 = 1$. Then discuss each of these points carefully below

- "multiplying by $a + bi$" is the same as rotating by some angle θ
- "multiplying by $c + di$" is the same as rotating by some angle ϕ
- "multiplying by $a+bi$ and then by $c+di$" should be the same as rotating by θ and then rotating by ϕ.
- "So multiplying $(a + bi) \cdot (c + di)$" should be the same as rotating by an angle of $\theta + \phi$.

(12) if you truly agree with all of those statements then you must think that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}.$$

Discuss why this must be.

(13) Finally, multiply the matrices on the left and then equate the entries. You should get some familiar formulas for trig functions.