FINITE FIELDS

- Recall that a field is a set that has operations $+$ and $\times$.
- There is a number 0 - the additive identity.
- And a number 1 - the multiplicative identity.
- We require that we can add and subtract and that nonzero elements have multiplicative inverses.
- Remember that we want the operations to be "closed" - in other words if we have x and 1 in our set then $x+1$ had better be in our set as well.
- We also require that fields obey the commutative, distributive and associative properties for $+$ and $\times$.

(1) Write down some examples of fields on the board - try to write down 3 infinite fields and 3 finite fields.

(2) A few weeks ago, you all wrote down a field with four elements. Was it $\mathbb{Z}/4\mathbb{Z}$? No! No! No! It wasn't! $\mathbb{Z}/4\mathbb{Z}$ is NOT A field. 2 doesn't have a multiplicative inverse in $\mathbb{Z}/4\mathbb{Z}$. If we want a field with 4 elements, we have to come up with DIFFERENT ways of adding and multiplying them.

What you wrote down was a set $\{0, 1, \alpha, \alpha + 1\}$ that satisfied the rules

$$1 + 1 = 0$$

$$\alpha(\alpha + 1) = 1$$

Expand out this rule and write it as an equation of the form some formula involving $\alpha = 0$. Use these rules to explain how to compute $\alpha^3(\alpha+1+1+1)$. Your answer should be one of $\{0, 1, \alpha, \alpha+1\}$.

What is α^4?

(3) As another example, consider the set $F = \{0, 1, 2, \beta, \beta+1, \beta+2, 2\beta, 2\beta+1, 2\beta+1\}$ with the rules

$$1+1+1=0$$
$$\beta^2 = -1$$

Use these rules to try and figure out what

$$-1$$

and

$$2 \cdot 2$$

and

$$\beta + (2\beta+1)(1-\beta)$$

and

$$\beta^3$$

are.

In fact, can you write down a multiplication table for these nine elements? Divide and conquer! I'd like each member of your group to understand how to do these computations? Is this a field?

(4) Let's try another example $F = \{0, 1, 2, \beta, \beta+1, \beta+2, 2\beta, 2\beta+1, 2\beta+1\}$ but with a different set of rules:

$$1+1+1=0$$
$$\beta^2+\beta+1=0$$

Is this a field? Write out your multiplication table!