THE INTEGERS

Work with a group of 2 or 3 students discussing the following problems. (Some of these problems appear on the homework this week.)

Some problems from today

We are going to learn a few abstract ideas today. **Our goal** is to learn how to deal with these abstract settings and gain some confidence. We will be dealing with a S and an operation we call $+$. This means that given two elements $s, t \in S$ we can assign to them a new element, called $s + t \in S$.

(1) I'll give you some examples of sets S along with some operations. Talk about these. Remember we are giving a "new" definition of $+$ in each case:
   (a) $S = \mathbb{N}$, $n + m :=$ usual sum.
   (b) $S = \mathbb{N}$, $n + m := n^m$
   (c) $S = \mathbb{Z}$, $n + m := 0$ always
   (d) $S = \mathbb{Z}$, $n + m := n - m$
   (e) $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $n + m := n + m$ but added like on a clock. E.g. $4 + 9 = 1$.
   (f) $S = \{people\}$, $x + y := x$
   (g) $S = \{people\}$, $x + y := \begin{cases} \text{the oldest child of } x \text{ and } y & \text{if } x \text{ and } y \text{ have a child} \\ \text{Abraham Lincoln} & \text{otherwise} \end{cases}$

   We say that the operation $+$ is associative if for all $s, t, u \in S$
   $$s + (t + u) = (s + t) + u.$$
   We say that the operation $+$ is commutative if for all $s, t \in S$
   $$s + t = t + s.$$

(2) Which of the sets and operations above are associative? Commutative?

(3) Let S and $+$ be a set with an addition rule that is associative and commutative. Can you define what you think it means to be an **additive identity**? Think carefully about this and discuss with your group. You definition should start "$z \in S$ is called an additive identity if ..."

(4) Which of the examples above have an additive identity? If they have one - how many do they have?

(5) Prove that if a set has an additive identity then there is only one.

(6) If S has an additive identity z, and $s \in S$ is an element then it's possible that s could have an **additive inverse**, which we denote $-s$. How would you define $-s$?

(7) Show that if an element $s \in S$ has an additive inverse then it is unique.

(8) Show that the additive inverse of $-s$ is s.

Looking Forward

- Before Wednesday's class, please (re)read Aaron Bertram's notes on the rational numbers- Section 1.2.

Warmup As a warmup, let's prove that there are infinitely many prime numbers! Think about the following proof.

Proof:

Assume that there are finitely many prime numbers $\{p_1, \ldots, p_k\}$. In other words imagine that that is all the prime numbers. We want to now get a **contradiction**. If we are successful, then that will mean our assumption had to be incorrect. This is how proof by contradiction works.

Ok, let's do it! Consider the number

$$N = p_1 \cdot p_2 \cdots p_k + 1.$$

In other words, N is the product of all the primes plus 1.

- Talk about why N is not divisible by p_1
- Talk about why N is not divisible by any p_i
- Talk about why this means N is not divisible by any prime.
- Talk about why this is a contradiction!

Some Critical Thinking

- Does this mean that N has to be prime? Test this: Try $(2) + 1$ and then $(2)(3) + 1$ and then $(2)(3)(5) + 1$ etc. Are they all prime?
- What if we modified the above proof and had $N = p_1 \cdot p_2 \cdots p_k + 2$. How would that change your answers above?
- What if we forced you to use $N = blah + 2$ in your proof, how could you fix your proof to still get a contradiction?