# Little Bit of Magic

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Hi everyone. I've just glanced through the homework and see lots of good things. I noticed that everyone thought that $x^4 + 1$ was prime in $\mathbb{R}[x]$. Well.

$$\Huge x^4 + 1 \text{ is not prime in } \mathbb{R}[x]$$

You'd better believe it. In fact - you can just check:

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1).$$

## 1 Now, why is that?

Well there are probably some devilish ways to see that it factors. But let's try to use some complex numbers. So if we want the roots of this:

$$x^4 + 1 = 0$$

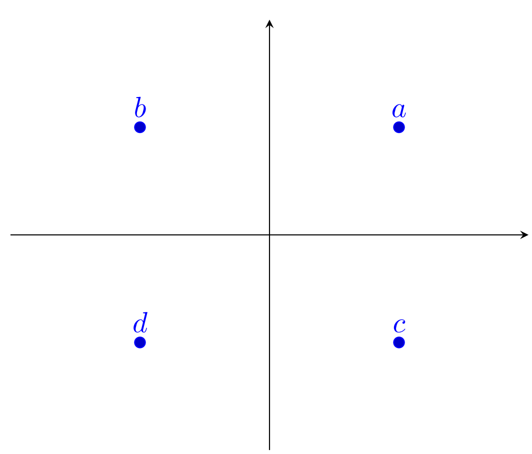
They should be the four roots of $-1$. In other words the 4 roots of $(1, 180^o)$. Ok, so those are:

$$(1, 45^o), (1, 135^o), (1, 225^o), (1, 315^o)$$

Let's plot those points. Now let's call these complex numbers $a, b, c, d$. We could write out their real and imaginary parts, but it'll be icky and probably involve the square root of 2. For now, let's just satisfy our selves that

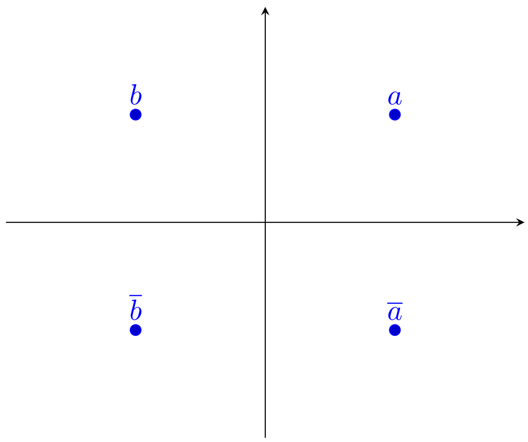
$$|a|^2 = 1, \quad |b|^2 = 1, \quad \text{and } \mathrm{Re}\ a = \sqrt{2}/2, \quad \mathrm{Re}\ b = \sqrt{2}/2.$$

This is pretty easy - right, the lengths of these things are all 1. And the real part of $a$ and $b$ is this number. Ok. Picture:



## 2 These things have symmetry

You should see that these points have some mega symmetry. There are some complex conjugates floating around. Let's do a better job of labeling:




Now we know that over $\mathbb{C}$:

$$x^4 + 1 = (x - a)(x - \overline{a})(x - b)(x - \overline{b}).$$

(Just multiplying the roots together). But look at the magic that happens when you multiply something like this out!

$$= (x^2 - ax - \overline{a}x + a\overline{a})(x^2 - bx - \overline{b}x + b\overline{b})$$
$$= (x^2 - (a + \overline{a})x + |a|^2)(x^2 - (b + \overline{b})x + |b|^2)$$

Hmm isn't $a + \overline{a}$ going to mean that the imaganary parts cancel out? Yes. And all we'll get is the real part twice! And isn't that $|a|^2 = 1$ from above? Yes. So

$$x^4 + 1 = (x^2 - 2(\text{Re } a)x + 1)(x^2 - 2(\text{Re } b)x + 1)$$
$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1).$$

Pretty cool!