THE NATURAL NUMBERS II

Work with a group of 2 or 3 students discussing the following problems. (Some of these problems appear on the homework this week.)

Some problems from today

(1) Find the gcd of 13 and 34 by factoring and also by using the Euclidean Algorithm

(2) Find the gcd of 12008 and 12 by using the Euclidean algorithm (you'll have to remember how to long divide by hand!)

(3) Let's try to understand the Euclidean algorithm in a few slightly more abstract examples. Suppose that a student applies the Euclidean algorithm to two numbers $a < b$ and obtains the following work:

$$\begin{gathered} b = aq + r_1 \\ \dots \\ \dots \\ 34 = 14 \cdot 2 + 6 \end{gathered}$$

then what is $gcd(a, b)$?

(4) Suppose that you see the following steps in someone's Euclidean algorithm for $a < b$:

$$\begin{gathered} b = aq_1 + 10 \\ a = 10q_1 + r_2 \\ \dots \end{gathered}$$

What does this information tell you about a and r_2? (I'm looking for some inequalities)

(5) Finally, in the same setup as above, you can say that $gcd(a, b) = gcd(a, 10)$. Why?

Now try some problems about divisibility:

(6) Do you think it is true that if a divides b and b divides c, then a divides c? Find a proof or a counterexample.

(7) Suppose that a and b both divide n. Is it true that $ab|n$? (That symbol means that ab divides n.) Find a proof or counterexample. What if we require a and b to be prime? Can you find any version of this that is true? :)

(8) Let T be the set of natural numbers that leave a remainder of 3 when divided by 4. Can you write a formula, say in terms of k that describes all the numbers in T? What happens if you multiply two of these numbers together? Do you get another element in that set? What if you multiply three together?...

Looking Forward

- From last time (and on hwk) I define X_n to be the set $\{1, 2, \dots, n\}$. Write down X_4 and all of its subsets. (Don't forget the empty set!) Prove by induction that there are in general 2^n subsets of X_n. Hint: exactly half of the subsets of X_{n+1} contain $n + 1$.
- Before Wednesday's class, please read Aaron Bertram's notes on the integers - Section 1.2.

Warmup As a warmup, we'll do another induction proof. But before we get started, let's review one very useful fact about inequalities: Suppose that a, b, c, d are natural numbers and

$$a < b, \quad c < d$$

then $ac < bd$.

Talk about this with your group - why does this make sense? I'm not looking for a proof, just some intuition

Now, using induction, prove that

$$2^n \leq 2n!$$

for all natural numbers n.

Remember - for induction, you need to prove that the result works for a base case ($n = 1$) and then assume that it is true for some k and then prove this means it is true for $k + 1$.