THE RATIONALS

Work with a group of 2 or 3 students discussing the following problems. (Some of these problems appear on the homework this week.)

Some problems from today

(1) Discuss what it means to be "well-defined." Explain why the following functions are NOT well-defined:

$f : \mathbb{Z} \to \mathbb{R}$, $f(n) = \sqrt{n}$.

$f : \mathbb{N} \to \mathbb{Z}$, $f(n) = \sqrt{n}$

$f : \mathbb{N} \to \mathbb{R}$, $f(n) =$ the number a such that $a^2 = n$.

$f : \mathbb{Q} \to \mathbb{Z}$, $f(\frac{a}{b}) = a + b$.

$f : \{\text{US states}\} \to \mathbb{Z}$, $f(x) =$ the age of the senator from state x.

It's important to understand that being well-defined is a bit complicated. There are lots of ways something can fail to be well-defined.

(2) Prove that multiplication of two rational numbers defined by

$$(\frac{a}{b})(\frac{c}{d}) = \frac{ac}{bd}$$

is a well defined operation.

(3) Does $\mathbb{Q}$ have an additive identity? Multiplicative identity? What elements of $\mathbb{Q}$ have a multiplicative inverse?

(4) What does it mean to say that a rational number is in lowest terms?

Definitions. Mathematics depends on making **precise** definitions. What I will want to see is that each group has a carefully written definition of each term. This might require a few iterations. That's ok!

You can assume that S is a set with operations $+$ and $\cdot$ that satisfy the following properties for all $s, t, u \in S$.

$$s + t = t + s, \qquad s + (t + u) = (s + t) + u$$

$$s(tu) = (st)u, \qquad st = ts, \qquad s(t + u) = st + su$$

(1) What are the names of each of these properties?

(2) Define what it means to be an **Additive Identity** in S.

(3) Define what it means to be an **Additive Inverse** for an element $s \in S$. (Hint: you'll probably need to assume that you have an additive identity)

(4) Define what it means to be a **Multiplicitive Identity** in S.

(5) Define what it means to be a **Multiplicitive Inverse** of an element $s \in S$.

Warning: You might very well be tempted to write things like 0 or 1 in the above. This is ultimately a good thing - it means you have good intuition as to what additive and multiplicative identities look like. But remember, our system might not even involve numbers! It might involve matrices, or numbers on the clock, or something even more abstract.

As an exercise, consider the clock-arithmetic system on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ where we can add and multiply (but always reducing like on a 12 clock) numbers. Does this set have any of the above properties? (Warning: the answer is not yes to all the above!)