Math 320 Linear Algebra
Assignment # 12

1. Consider
$$A = \begin{bmatrix} 1 & -2 \\ 1 & -3 \\ 4 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 1 & y \\ x & 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 10 & -3 & 2 \\ 16 & -5 & 1 \\ -2 & 2 & z \end{bmatrix}.$$
   (a) Find x, y and z so that $AB = C$.

   (b) Find BA

2. Consider:
$$A^{-1} = \begin{bmatrix} 3 & 2 & -1 \\ 1/2 & 0 & 4 \\ -2 & 1 & -2 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} 3/2 & -2 & 1 \\ 3 & 0 & -4 \\ 0 & 1 & -2 \end{bmatrix}.$$
   Suppose $C = AB$. Find C^{-1}.

3. In the last homework you should that if $T_1 : W \to V$ and $T_2 : V \to U$ are linear transformations then

$$\begin{aligned} \mathscr{R}(T_2 \circ T_1) &\le \mathscr{R}(T_2) \\ \mathscr{N}(T_1) &\le \mathscr{N}(T_2 \circ T_1). \end{aligned}$$

You may use those results in the following. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.

(a) Show that $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.

(b) Show that $\operatorname{rank}(AB) \le \operatorname{rank}(B)$.

(c) Show that $\operatorname{rank}(AB) \le \min(m, n, p)$. (Note $\min(m, n, p)$ is the smallest of the three numbers m, n and p. That is show $\operatorname{rank}(A) \le m$, $\operatorname{rank}(A) \le n$, and $\operatorname{rank}(A) \le p$.)

(d) Suppose $\vec{w}, \vec{v}$ are non-zero elements of $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. Let $A = \vec{w}\vec{v}^T$.

i. What is the $\operatorname{rank}(A)$?

ii. Let $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix}$. Calculate A in this case.

4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define $\det(A) = ad - bc$.

   (a) Show that if $\det(A) = 0$ then A is singular. (Hint two cases $a \neq 0$ and see homework# 8 and $a = 0$.)

   (b) Conversely show that if $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

5. Consider:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 & 4 \\ 1 & 2 & 3 & 5 & -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 & 5 & -1 & 2 \\ 0 & 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   B is the output of doing Gaussian elimination on A.

   (a) Find elementry matrices $E_1, E_2, \ldots, E_r$ such that:

$$E_r E_{r_1} \ldots E_2 E_1 A = B$$

(b) Find an invertible matrix C such that $CA = B$.

(c) Do the matrix multiplication to show that indeed $CA = B$.

(d) Find $E_1^{-1}, E_2^{-1}, \ldots, E_r^{-1}$ (that is find the inverse of each of the elementary matrices found about).

(e) Use the previous part to C^{-1}.

(f) Verify by matrix multiplication that $A = C^{-1}B$.