Math 320 Linear Algebra
Assignment # 10

1. Each of the following you may assume are linear transformation. For each find a basis for both $\mathscr{R}(T)$ and $\mathscr{N}(T)$. Find rank(T) and nullity(T). Verify the rank and nullity theorem for this particular example.
   (a) $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^3$ defined by:
   $$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-b \\ c \end{bmatrix}.$$
   (b) Let $T : P_2 \to \mathbb{R}^3$ defined by:
   $$T(ax^2 + bx + c) = \begin{bmatrix} a+b \\ a+c \\ a \end{bmatrix}$$
   (c) $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T(\vec{v}) = A\vec{v}$ where:
   $$A = \begin{bmatrix} 3 & 3 & 1 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$
   (d) $T : P_2 \to \mathbb{R}$ defined by $T(p(x)) = \int_0^1 p(x)$.
   (e) $T : P_n \to P_n$ defined by $T(p(x)) = p'(x)$.

2. For each of the following you may assume that $\mathscr{B}$ is a basis for V. Find the following:
   (a) Let $V = \mathbb{R}^3$, $\mathscr{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$. Find $\mathrm{Rep}_{\mathscr{B}}\left(\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}\right)$.
   (b) Let $V = \mathbb{R}^{2\times 2}$, $\mathscr{B} = \left(\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\right)$. Find $\mathrm{Rep}_{\mathscr{B}}\left(\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\right)$.
   (c) Let $V = P_2$, $\mathscr{B} = \left(x^2 + x - 2, x^2 - 3x + 4, x^2 + 1\right)$. Find $\mathrm{Rep}_{\mathscr{B}}\left(x^2\right)$.

3. Suppose W, V and U are vector spaces and $f : W \to V$, $g : V \to U$. Prove:
   (a) If f and g are one-to-one then $g \circ f$ is one-to-one.
   (b) If f and g are linear transformations then $g \circ f$ is a linear transformation.

4. Suppose that W and V are vector spaces and $T : W \to V$ is a linear transformation. Show that $\mathscr{N}(T) = \{\vec{w} \in W : T(\vec{w}) = \vec{0}_V\}$ is indeed a subspace of W.