Math 320 Linear Algebra
Assignment # 13

1. In this problem we are going to an important theorem. Let V and W be vectors spaces and $T : V \to W$ be an isomorphism (a $1-1$ and onto linear transformation). Furthermore let $\mathscr{B} = \{\vec{v_1}, \vec{v}_2, \ldots, \vec{v}_r\} \subseteq V$ and $\mathscr{D} = \{T(\vec{v_1}), T(\vec{v}_2), \ldots, T(\vec{v}_r)\} \subseteq W$.
   (a) If $\mathscr{B}$ is linearly independent in V then $\mathscr{D}$ is linearly independent in W. Here is a short video that will help with this part.
   (b) If $\mathscr{B}$ is spans V then $\mathscr{D}$ spans W. Here is a short video that will help with this part.
   (c) If $\mathscr{B}$ are a basis for V then $\mathscr{D}$ are a basis for W.
   (Hint: this should be very straight forward from the previous part)

2. Consider $V = \mathbb{R}^{2\times 2}$ and let

$$\mathscr{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

   (a) What is $\dim V$
   (b) Show that $\mathscr{B}$ is linearly independent.
   (c) Why can you conclude from above that $\mathscr{B}$ is a basis for V?
   (d) Let

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

   Find $[A]_{\mathscr{B}}$. (Hint: Your answer should be an element of $\mathbb{R}^4$. Remember the order of $\mathscr{B}$ matters.)

3. Let $T : \mathbb{R}^{2\times 2} \to P_2$ be defined by:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = bx^2 + bx + c$$

   (Hint: Only the first problem should require much in the way of work)
   (a) Find a basis for $\ker(T)$.
   (b) Find $\dim(\ker(T))$.
   (c) Find $\dim(\mathrm{Rg}(T))$
   (d) Find a basis for $\mathrm{Rg}(T)$
   (e) Is T onto?

4. Let

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & -1 \\ 3 & 2 & -2 & 0 & 0 \\ -3 & -1 & 3 & 5 & 1 \\ 0 & 0 & -2 & 0 & 3 \\ -3 & -1 & 1 & 0 & 0 \end{bmatrix}$$

   (a) Find $\det(A)$ by expanding on the first row. Here is a short video that teaches you how to do that.

(b) Find $\det(A)$ by expanding on a different row or column. short video that teaches you how to do that.

5. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ -2 & -2 \end{bmatrix}$. Find the following:

   (a) $\det(A)$

   (b) $\det(B)$

   (c) AB

   (d) $\det(AB)$

   (e) Show $\det(A)\det(B) = \det(AB)$.