Math 320 Linear Algebra
Assignment # 6

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by:

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ 3c \end{bmatrix}$$

   (a) Find:

$$5 \cdot T\left(\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right)$$

   (b) Find:

$$T\left(5 \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right)$$

   (c) Are the previous two parts equal? Does that guarantee that T is linear?

   (d) Prove that T is linear by using the one step verification.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-2b \\ 3b \\ 2a-b \end{bmatrix}$$

   (a) Either prove that T is linear by finding A such that $T(\vec{v}) = A\vec{v}$, or show that T is not linear with a specific example.

   (b) Find $\vec{v}$ such that $T(\vec{v}) = \begin{bmatrix} 10 \\ -9 \\ 11 \end{bmatrix}$, or explain why no such $\vec{v}$ exists.

   (c) Find $\vec{v}$ such that $T(\vec{v}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, or explain why no such $\vec{v}$ exists.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ a^2+1 \\ -b \end{bmatrix}$$

   (a) Either prove that T is linear by finding A such that $T(\vec{v}) = A\vec{v}$, or show that T is not linear with a specific example.

   (b) Find $\vec{v}$ such that $T(\vec{v}) = \begin{bmatrix} 10 \\ -9 \\ 11 \end{bmatrix}$, or explain why no such $\vec{v}$ exists.

   (c) Find $\vec{v}$ such that $T(\vec{v}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, or explain why no such $\vec{v}$ exists.

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be general elements of $\mathbb{R}^3$. Also let $c, d \in \mathbb{R}$ be general.

Then:

$$T(c\vec{u} + d\vec{v})$$

$$= T\left(\begin{bmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \\ cu_3 + dv_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} cu_1 + dv_1 - (cu_2 + dv_2) \\ 3(cu_3 + dv_3) \end{bmatrix}$$

$$= \begin{bmatrix} c(u_1 - u_2) + d(v_1 - v_2) \\ 3cu_3 + 3dv_3 \end{bmatrix}$$

Also

$$cT(\vec{u}) + dT(\vec{v})$$

$$= c\begin{bmatrix} u_1 - u_2 \\ 3u_3 \end{bmatrix} + d\begin{bmatrix} v_1 - v_2 \\ 3v_3 \end{bmatrix}$$

$$= \begin{bmatrix} c(u_1-u_2) \\ 3cu_3 \end{bmatrix} + \begin{bmatrix} d(v_1-v_2) \\ 3dv_3 \end{bmatrix}$$

$$= \begin{bmatrix} c(u_1-u_2) + d(v_1-v_2) \\ 3cu_3 + 3dv_3 \end{bmatrix}$$

So $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

since $\vec{u}, \vec{v}, c, d$ are general

it follows that T is linear.