Problems from Assignment 10

1. In class we are using the fact that if $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$ then:

$$\overline{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Let's remind ourselves why this is true (this is actually a different proof then the one I gave in Math 350).

For this problem you may use the fact that the density for the normal distribution is a density. That is you may use the fact (you don't need to reprove it) that if $\sigma > 0$ then:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}\, du = 1.$$

You may also use the fact (this is easy to prove and was certainly done in your probability class) that if $X \sim N(\mu, \sigma^2)$ then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.

Remember that the moment generating function for Y is:

$$m_Y(t) = \mathrm{E}(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f_Y(y)\, dy$$

(a) Let $Y \sim N(0, \sigma^2)$ show that

$$m_Y(t) = e^{\frac{t^2\sigma^2}{2}}.$$

Hint: Complete the square.

(b) Let $X \sim N(\mu, \sigma^2)$ find $m_X(t)$. Hint: $(X - \mu) \sim N(0, \sigma^2)$.

(c) Suppose $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ with X_1 and X_2 being independent. Use the previous parts to show that $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(d) Prove by induction that if for $1 \le i \le n$, $X_i \sim N(\mu_i, \sigma_i^2)$ with the random variables being independent then:

$$\sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

(e) Prove if $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$ then:

$$\overline{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

2. Suppose $Z_1, Z_2, Z_3 \overset{\text{iid}}{\sim} N(0, 1)$ and define:

$$\begin{aligned} X_1 &= Z_1 + 2Z_2 - 2Z_3 \\ X_2 &= 2Z_1 + Z_2 + 2Z_3 \\ X_3 &= 6Z_1 - 6Z_2 - 3Z_3 \end{aligned}$$

(a) What are the distributions of X_1, X_2, X_3 and the relationship between them. (Hint: Use Fisher's Theorem)

(b) What is the distribution of $X_1 + X_2 + X_3$?

(c) Find $\mathrm{P}(X_1 + X_2 + X_3 \le 3)$.

(d) Find $a, b, c \in \mathbb{R}$ such that $aX_1^2 + bX_2^2 + cX_3^2 \sim \chi_3^2$.