1. Suppose $f \in \mathcal{R}[a,b]$ and $m, M \in \mathbb{R}$.

   (a) Suppose $m \le f(x) \le M$ for all $x \in [a,b]$ show $m(b-a) \le \displaystyle\int_a^b f(x) \le M(b-a)$.

   (b) Suppose $|f(x)| \le M$ for all $x \in [a,b]$ show $\left|\displaystyle\int_a^b f\right| \le M(b-a)$

   (c) Suppose $|f(x)| \le M$ for all $x \in [a,b]$ and $\alpha, \beta \in [a,b]$ show $\left|\displaystyle\int_\alpha^\beta f\right| \le M|\beta - \alpha|$

2. For each of the following determine (with proof) if f is Lipschitz continuous on the given domain.

   (a) $f(x) = x^2$ on $[0,1]$

   (b) $f(x) = x^2$ on $(1,\infty)$

   (c) $f(x) = \sqrt{x}$ on $[1,2]$

   (d) $f(x) = \sqrt{x}$ on $(1,\infty)$

3. Let $f(x) = \dfrac{1}{x}$. For any positive number $a \in \mathbb{R}$. Do the following:

   (a) Prove $f \in \mathcal{R}[\min(a,1), a+1]$

   (b) Define $\ln(x) = \displaystyle\int_1^x f$ (this is all you know about natural log so don't sneak rumors you have heard about it). Prove $\ln(x)$ is continuous on $(0,\infty)$.

   (c) Prove that $\ln(x)$ is increasing on $(0,\infty)$

   (d) Let $a > 0$ be fixed for now. Let $g(x) = \ln(x) + \ln(a)$ and $h(x) = \ln(ax)$. Show $h'(x) = g'(x)$ for all $x \in (0,\infty)$.

   (e) Argue that in the above $h(x) = g(x)$ for all $x \in (0,\infty)$

   (f) Argue from the above that for all $a, b \in (0,\infty), \ln(ab) = \ln(a) + \ln(b)$.

   (g) Prove that $\ln(x)$ is injective $(1-1)$ on $(0,\infty)$

   (h) Let $B = \ln((0,\infty))$ be the image of f and let $\exp(x)$ be the inverse function of $\ln(x)$ on $(0,\infty)$ (one exists since $\ln(x)$ is injective). So $\exp : B \to (0,\infty)$. Prove that $\exp(x)$ is continuous

   (i) Prove that $\exp(x)$ is increasing.

   (j) Prove for all $a, b \in B$, $\exp(a+b) = \exp(a)\exp(b)$.