1. Show $\ln(2) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$

2. More fun with $\exp(x)$ and $\ln(x)$. For this problem you may use any of the results in the previous homework (whether you proved it or not).

   (a) Let $q \in \mathbb{Q}$ and $a > 0$ show that $\ln(a^q) = q\ln(a)$.

   (b) Let $q \in \mathbb{Q}$ show that $\exp(q) = e^q$.

   (c) Suppose $a > 0$ define $a^x = \exp(x\ln(a))$. We (I mean in real analysis I) have previously defined a^q when $q \in \mathbb{Q}$, show this definition is consistent with that definition.

   (d) Let $a > 0$ and define $f(x) = a^x$, show that f is differentiable on $(-\infty, \infty)$ and find $f'(x)$.

   (e) Following what we did in class with Taylor's theorem, to show for any $b \in \mathbb{R}$:

   $$\exp(b) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{b^n}{n!}$$

   Hint: I assume that in Math 360 you proved that for all $x \in \mathbb{R}$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. In any case you may use this fact.

   (f) Show $\frac{8}{3} \leq e$.

3. For all $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ where:

   $$f_n(x) = \begin{cases} n^2x & \text{if } x \in [0, \frac{1}{n}] \\ 2n - n^2x & \text{if } x \in (\frac{1}{n}, \frac{2}{n}) \\ 0 & \text{if } x \in [\frac{2}{n}, 1] \end{cases}$$

   (a) Draw f_n for a $n = 1, 10, 100$.

   (b) Find f such that $f_n \to f$ pointwise.

   (c) Does $\int_0^1 f_n \to \int_0^1 f$?

   (d) Are f_n and f continuous?

   (e) Does f_n converge to f uniformly?