Homework Due on Feb 17, 2015

1. Finish the proof that if $\alpha, \beta, \gamma \in [a, b]$ and $f \in \mathcal{R}[a, b]$, then:

$$\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f$$

2. Suppose $f, F : [a, b] \to \mathbb{R}$ with F differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$ and $f \in \mathcal{R}[a, b]$. Suppose $\alpha, \beta \in [a, b]$ show:

$$\int_\alpha^\beta f = F(\beta) - F(\alpha)$$

3. Let $F : [0, 3] \to \mathbb{R}$ defined by:

$$F(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ x - 1 & \text{if } 1 < x < 2 \\ 1 & \text{if } 2 \leq x \leq 3. \end{cases}$$

   (a) Prove that $F \in c([0, 3])$

   (b) Find an $f : [0, 3] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all but a finite set.

   (c) Find $\displaystyle\int_0^3 f$.

   (d) Find $\displaystyle\int_0^3 F$.

4. Suppose $f \in \mathcal{R}[a, b]$ and $m, M \in \mathbb{R}$.

   (a) Suppose $m \leq f(x) \leq M$ for all $x \in [a, b]$ show $m(b - a) \leq \displaystyle\int_a^b f(x) \leq M(b - a)$.

   (b) Suppose $|f(x)| \leq M$ for all $x \in [a, b]$ show $\left|\displaystyle\int_a^b f\right| \leq M(b - a)$

   (c) Suppose $|f(x)| \leq M$ for all $x \in [a, b]$ and $\alpha, \beta \in [a, b]$ show $\left|\displaystyle\int_\alpha^\beta f\right| \leq M|\beta - \alpha|$

5. For each of the following determine (with proof) if f is Lipschitz continuous on the given domain.

   (a) $f(x) = x^2$ on $[0, 1]$
   (b) $f(x) = x^2$ on $(1, \infty)$
   (c) $f(x) = \sqrt{x}$ on $[1, 2]$
   (d) $f(x) = \sqrt{x}$ on $(1, \infty)$