Homework Due on February 19, 2015

1. Let $f(x) = \dfrac{1}{x}$. For any number $a \in \mathbb{R}^+$. Do the following:

   (a) Prove $f \in \mathcal{R}[\min(a, 1), a + 1]$

   (b) Define $\ln(x) = \displaystyle\int_1^x f$ (this is all you know about natural log so don't sneak rumors you have heard about it). Prove $\ln(x)$ is continuous on $(0, \infty)$.

   (c) Prove that $\ln(x)$ is increasing on $(0, \infty)$

   (d) Let $a > 0$ be fixed for now. Let $g(x) = \ln(x) + \ln(a)$ and $h(x) = \ln(ax)$. Show $h'(x) = g'(x)$ for all $x \in (0, \infty)$.

   (e) Argue that in the above $h(x) = g(x)$ for all $x \in (0, \infty)$

   (f) Argue from the above that for all $a, b \in (0, \infty), \ln(ab) = \ln(a) + \ln(b)$.

   (g) Prove that $\ln(x)$ is injective $(1 - 1)$ on $(0, \infty)$

   (h) Let $B = \ln((0, \infty))$ be the image of f and let $\exp(x)$ be the inverse function of $\ln(x)$ on $(0, \infty)$ (one exists since $\ln(x)$ is injective). So $\exp : B \to (0, \infty)$. Prove that $\exp(x)$ is continuous

   (i) Prove that $\exp(x)$ is increasing.

   (j) Prove for all $a, b \in B$, $\exp(a + b) = \exp(a)\exp(b)$.

2. Suppose $f \in b[a, b]$ then show that $f \in \mathcal{R}[a, b]$ if and only if for all $\epsilon > 0$ there exists a partition $\mathcal{P} = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ such that if $\{t_i\}$ and $\{q_i\}$ are any tags for $\mathcal{P}$ then:

$$\sum_{i=1}^{n} |f(t_i) - f(q_i)| (x_i - x_{i-1}) < \epsilon.$$