Homework Due on March 03, 2015

1. For this problem you may use any of the results in the previous homework (whether you proved it or not). Remember for $a > 0$ and $x \in \mathbb{R}$, we defined $a^x = \exp(x \ln(a))$.

   (a) Show that if $a > 0$ and $x, y \in \mathbb{R}$ show that $a^{x+y} = a^x a^y$.

   (b) Show that if $a > 0$ and $x \in \mathbb{R}$ then $\ln(a^x) = x \ln(a)$.

   (c) Show that if $a > 0$ and $x, y \in \mathbb{R}$ show that $a^{xy} = (a^x)^y$.

   (d) Let $a \in \mathbb{R}$ and define $f(x) = x^a$. Show that f is differentiable on $(0, \infty)$, and find $f'(x)$.

2. For each $n \in \mathbb{N}$ let $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Show that for $M > 0$, $f_n \to \exp(x)$ uniformly on $[-M, M]$.

3. For all $n \in \mathbb{N}$ define $f_n : [0, 1] \to \mathbb{R}$ where:

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}] \\ 2n - n^2 x & \text{if } x \in (\frac{1}{n}, \frac{2}{n}) \\ 0 & \text{if } x \in [\frac{2}{n}, 1] \end{cases}$$

   (a) Draw f_n for a $n = 1, 10, 100$.

   (b) Find f such that $f_n \to f$ pointwise.

   (c) Does $\int_0^1 f_n \to \int_0^1 f$?

   (d) Are f_n and f continuous?

   (e) Does f_n converge to f uniformly?