Homework Due on April 14, 2015

1. Suppose $\vec{x} \in \mathbb{R}^n$ show:
   (a) $||\vec{x}||_\infty \leq ||\vec{x}||_1 \leq n||\vec{x}||_\infty$
   (b) $||\vec{x}||_\infty \leq ||\vec{x}||_2 \leq \sqrt{n}||\vec{x}||_\infty$
   (c) Show that a set is open in $(\mathbb{R}^n, ||\cdot||_1)$ if and only if it is open in $(\mathbb{R}^n, ||\cdot||_2)$ if and only if it is open in $(\mathbb{R}^n, ||\cdot||_\infty)$

2. Let $a \in \mathbb{R}^\infty$ with $\mathbf{a} = \{a_n\}_{n=1}^\infty$

$$a_n = \begin{cases} 5^{-n} & \text{if } n \text{ is odd} \\ 2^{-n} & \text{if } n \text{ is even} \end{cases}$$

   Find:
   (a) $||\mathbf{a}||_1$
   (b) $||\mathbf{a}||_2$
   (c) $||\mathbf{a}||_\infty$

   Hint: If a series converges absolutely then the series can be summed in any order.

3. Show that $(L^1, ||\cdot||_1)$ is a normed vector space. Remember that L^1 consists of equivalence classes so that is why $||f||_1$ is 0 only when f is 0 in the sense that it is in the same equivalence class as 0.

4. Show $\frac{\sqrt{2}}{\sqrt{\pi}}\sin(x)$ and $\frac{\sqrt{2}}{\sqrt{\pi}}\cos(x)$ form an orthonormal basis for a finite dimensional subspace of $L^2([0, \pi])$. (You might have to look up some linear algebra.)