Math 370 Number Theory
Assignment # 12

1. Verify by direct computation that $3087 = \sum_{d|3087} \phi(d)$.

2. Let $F(n) = \sum_{d|n} \sigma(d)$. Find:
   (a) $F(10)$
   (b) $F(79)$
   (c) $F(27)$
   (d) $F(21330)$ (Hint $21330 = 10 \cdot 79 \cdot 27$).
   (e) Does $F(27) = F(3)F(9)$?

3. For $n \in \mathbb{N}$ let $\tau(n)$ be the number of positive divisors of n (including 1 and n).
   (a) Show that τ is arithmetic.
   (b) Find $\tau(p^k)$ where p is prime and $k \in \mathbb{N}$.
   (c) Find a formula for $\tau(n)$ based on the factorization of n.
   (d) Give a simple description of all n such that $\tau(n)$ is odd and prove your answer.
   (e) Answer this puzzle my dad recently sent me. He heard it on NPR's *Car Talk* years ago and often tried to find the answer for particular numbers. He recently found a general solution can you?

   > RAY: Imagine, if you will, that you have a long, long corridor that stretches out as far as the eye can see. In that corridor, attached to the ceiling are lights that are operated with a pull cord. There are gazillions of them, as far as the eye can see. Let's say there are 20,000 lights in a row. They're all off. Somebody comes along and pulls on each of the chains, turning on each one of the lights. Another person comes right behind, and pulls the chain on every second light.
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   > TOM: Thereby turning off lights 2, 4, 6, 8 and so on.
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   > RAY: Right. Now, a third person comes along and pulls the cord on every third light. That is, lights number 3, 6, 9, 12, 15, etcetera. Another person comes along and pulls the cord on lights number 4, 8, 12, 16 and so on. Of course, each person is turning on some lights and turning other lights off.
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   > If there are 20,000 lights, at some point someone is going to come skipping along and pull every 20,000th chain.
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   > When that happens, some lights will be on, and some will be off. Can you predict which lights will be on?

4. Suppose n is composite and not a Carmichael number.
   (a) Show that at least half of all numbers a such that $\gcd(a, n) = 1$ are Fermat witnesses. (Hint: Look how we showed it for Euler witnesses).
   (b) Show that a has at least $n - \frac{\phi(n)}{2}$ Fermat winesses.