Additional Problem Assignment 11

These problems are to be worked out by yourself. These are the level of difficulty of the exam so if you are going to do well on the exam you need to be able to do these problems alone. If you really need to ask someone for help, ask them a different question than the exact one asked. For example you can ask someone "Can you remind me how to find the subgroups of $\mathbb{Z}_n$?", but you may not ask them "What is a proper subgroup of $\mathbb{Z}_7$ with at least 5 elements?"

Before the exam on Thursday, focus on solving the problems (or mentally saying the solution if you don't feel you need to write it down) rather than writing up formal solutions to be turned in. You will only be turning in a (proper) subset of the problems assigned.

1. Give the definition of each of the following. You should be able to do this without looking at your book or notes. If you can't, look at your book or notes and then try again more than an hour later.
   (a) A binary operation (Jennifer)
   (b) An associative binary operation (Joseph, David, Griselda, Tony, William)
   (c) A commutative binary operation (Emma, Kelly, Katie, Beau, Chris)
   (d) An identity of a binary operation (Duggan, Lily, William, Anna)
   (e) The inverse of an element for a binary operation that has an identity (Duggan, Lily, Chris, Griffin)
   (f) A group (Emma, Kelly, Katie, Beau, Griselda)
   (g) An abelian group (Scott)
   (h) A subgroup of a group (Emma, Larissa, Katie, Bryce, David, Jennifer, Griffin)
   (i) The center of a group (Alexander, Scott, Anna)
   (j) The centralizer of a group (Griselda, Chris, Tony, Anna)
   (k) The group $U(n)$ where $n \in \mathbb{N}$. (Kelly, Bryce, Scott)
   (l) The order of a group (Alexander, Larissa, Beau, Griffin)
   (m) The order of an element of a group. (Joseph, David, Tony, William)
   (n) A cyclic group (Bryce, Jennifer)
   (o) Two other important definitions that are not listed above (there will be at least one on the exam). (Duggan, Joseph, Alexander, Larissa, Lily)

2. Find an example of each of the following (with an argument that the example works), or argue that no such example exists.
   (a) A binary operation on the set $\{a, b, c\}$ that is commutative but not associative (write a table for the operation) (David)
   (b) A group where every element is an involution. (Duggan, Emma, Scott, Anna)
   (c) A finite non-abelian group (Duggan, Joseph, Alexander, Larissa, Lily, Beau)
   (d) An element in $C\left(\left[\begin{smallmatrix} 8 & 3 \\ 1 & 2 \end{smallmatrix}\right]\right)$ other than the identity or the matrix itself (where we are working in $GL(2, \mathbb{R})$. (Kelly, Alexander, Griselda, Tony, Griffin)
   (e) An element in $\langle\left[\begin{smallmatrix} 8 & 3 \\ 1 & 2 \end{smallmatrix}\right]\rangle$ other than the identity or the matrix itself (where we are working in $GL(2, \mathbb{R})$. (Beau, Griselda, William, Anna)
   (f) Two elements of $SL(2, \mathbb{R})$ that do not commute. (Emma, Kelly, Larissa, Katie, Lily, Chris)
   (g) Two elements of $U(45)$ that do not commute. (Emma, Joseph, Larissa, Beau, Tony, Anna)

(h) A non-identity involution in $U(1897432890)$ (Alexander, Griffin)

(i) A proper subgroup of $\mathbb{Z}_{12}$ with at least 5 elements (a proper subgroup is a subgroup that is not the whole group). (Duggan, Katie, Bryce, Griselda, Tony)

(j) A proper subgroup of $\mathbb{Z}_7$ with at least 5 elements. (Larissa, Scott)

(k) A non-trivial proper subgroup of $U(8)$. (The trivial subgroup contains just the identity, so a non-trivial subgroup must contain more than the identity). (Joseph, Kelly, Katie, Lily, Scott, Jennifer, William, Griffin)

(l) An element of D_4 whose centralizer is a non-trivial proper subgroup. (Bryce, David, Jennifer, Chris)

(m) A subgroup of $U(8)$ that is not cyclic. (Bryce, Beau, David, Jennifer, Tony)

(n) A subgroup of $U(10)$ that is not cyclic. (Lily, David, Anna)

(o) A generator of $< 12 >$ in Z_{30} other than 12. (Kelly, Griselda, Scott, Jennifer, Chris, William)

(p) Two elements of order 8 in $\mathbb{Z}_{24}$. (Joseph, Alexander, Katie, Bryce, Chris, Griffin)

(q) Three elements of order 9 in $\mathbb{Z}_{25}$. (Duggan, Emma, William)

3. Perform the following calculations or explain why the calculation can't be done.

(a) $\left[\begin{smallmatrix} 3 & 1 \\ 2 & 5 \end{smallmatrix}\right]^{-1}$ in $GL(2, \mathbb{R})$. (Emma, Alexander, Griselda)

(b) $\left[\begin{smallmatrix} 3 & -1 \\ -6 & 2 \end{smallmatrix}\right]^{-1}$ in $GL(2, \mathbb{R})$. (Duggan, Lily, William, Griffin, Anna)

(c) $(a^2b^3a^{-4})^{-1}$ where a, b are elements in a group. (Kelly, Bryce, David, Scott, Tony)

(d) $(a^2b^3a^{-4})^{-1}$ where a, b are elements in an abelian group. (Emma, Joseph, Kelly, Larissa, Beau, Chris, Griffin)

(e) HR_{90} in D_4. (Bryce, Beau, David, Scott, Anna)

(f) $(18)(22)$ in $U(75)$. (Duggan, Alexander, Katie, Beau, Griselda, Jennifer)

(g) $(7)^{-1}$ in $U(50)$. (Katie, Lily, Scott, Chris)

(h) $(7)^5$ in $U(9)$. (Joseph, Kelly, Larissa, Jennifer, Tony, Anna)

(i) $< 3 >$ in $U(16)$. (Duggan, Alexander, Lily, Jennifer, Chris, William)

(j) $|8|$ in $\mathbb{Z}_{16}$. (Emma, Katie, Griffin)

(k) $|a^{12}|$ where $|a| = 8$. (Larissa, David, Griselda)

(l) Find all the generators of Z_{17}^*. (Joseph, Bryce, Tony, William)

4. Prove each of the following:

(a) Suppose G is an abelian group. If $H = \{a \in G : a^3 = e\}$ then $H \leq G$. (Duggan, Joseph, Larissa, David, Jennifer)

(b) In a group G with $a, b \in G$, if $a \neq b$ then either $a^4 \neq b^4$ or $a^5 \neq b^5$. (Griselda, Chris, William, Anna)

(c) Suppose G is a group and $x, y \in G$. $xyx^{-1} \in C(x)$ if and only if $y \in C(x)$. (Katie, Bryce, Lily, Griffin)

(d) Let a be an element of a group G. For all $k \in \mathbb{Z}$, $|a^k| \leq |a|$. (Emma, Kelly, Alexander, Beau, Scott, Tony)