Additional Problems Assignment 28

Here are some more sample problems to get you ready for the final. Note, there is a lot more to do then these problems. Make sure that you again do the reviews for exam 1 and exam 2, plus try to solve (without looking at solutions) all the problems from exam 1 and exam 2. Also look at the homework problems and try to solve as many as possible. Last but not least make sure look at the "NC" problems, since one of these problems will be on the final.

1. In each of the section not in the first two exams (Chapters 10,12-16) write down each of the important definitions and know them. Here are some obvious ones:
   (a) Group Homomorphisms
   (b) Ring
   (c) Field
   (d) Integral domain
   (e) Units of a ring
   (f) The polynomial ring $R[x]$ where R is a ring
2. Find an example of the following or explain why it is not possible.
   (a) A ring without unity.
   (b) An element of a ring that is not a zero divisor or a unit.
   (c) An integral domain that is not a field.
   (d) A non-trivial homomorphism from $\mathbb{Z}_{27}$ to $\mathbb{Z}_3$
   (e) A non-trivial homomorphism from $\mathbb{Z}_{27}$ to $\mathbb{Z}_{13}$
3. Perform the following calculations or explain why the calculation can't be done.
   (a) Find $q(x)$ and $r(x)$ such that $f(x) = g(x)p(x) + r(x)$ with $f(x) = 4x^4 + 3x^2 + x + 1$, $g(x) = 2x^2 + 6x + 3$ and $\deg(r(x)) < \deg(g(x))$ inside the polynomial ring:
      i. $\mathbb{Z}_7[x]$.
      ii. $\mathbb{Z}_{11}[x]$.
      iii. $\mathbb{Q}[x]$.
   (b) Let $f(x) = x^2 - 1 \in \mathbb{Z}_8[x]$.
      i. Find all the roots of $f(x)$
      ii. At first glance this seems to be a counterexample to something we proved on the last day. What result is this and why is it not a counterexample?
      iii. Explain why this shows that $U(8) \cong V_4$.
   (c) Consider the group $G = U(M_2(\mathbb{Z}_2)) = GL_2(\mathbb{Z}_2)$.
      i. Find all the elements of the group.
      ii. What familiar group is this group isomorphic to?
4. Prove each of the following:
   (a) Let $G = \{\left[\begin{smallmatrix} a & a \\ a & a \end{smallmatrix}\right] : a \in \mathbb{R}, a \neq 0\}$. Show that G is a group under matrix multiplication. (Hint: This not a hard problem on it face, but it is a little bit different then what your are used to, so there a lot of opportunities to go down the wrong path out of muscle memory.
   (b) Suppose F is a field and $|F| = 16$, show $F^* = U(F)$ is cyclic.

5. **Extra Credit:** As mentioned in class (and on our finite fielder uniforms) there is a field, $\mathbb{F}_9$ with 9 elements. Give the addition and multiplication table of the field.

   Hints:

   1. First let $2 = 1 + 1$ and then show that $1 + 1 + 1 = 0$.
   2. In class we proved that $U(\mathbb{F}_9)$ is cyclic. Argue that it has a generator a different from $0, 1, 2$.
   3. Show that the elements $0, 1, 2, a, 2a, a^2, 2a^2, a^3, 2a^3$ are all different and hence are the nine elements.
   4. Now make the table.