Additional Problem Assignment 3

Please do each of these problems on a separate sheet of paper, but not necessarily for each part of a problem.

For these problems let $\mathcal{M} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}$, that is the set of 2x2 matrices with integer entries. In each of the cases below you may assume without proof that the given binary operator is indeed a binary operator.

1. In Example 6, the book explains that $< M, + >$ is a group where, $+$ is regular matrix addition. That is:
$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}.$$
   (a) Calculate:
$$\begin{bmatrix} 7 & -2 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -3 & 5 \end{bmatrix}.$$
   (b) Using the fact that regular integer addition is associative to show that matrix addition is associative (in this 2x2 case).

2. Define a binary operation $\oplus_1$ on $\mathcal{M}$ by:
$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \oplus_1 \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + d_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + a_2 \end{bmatrix}.$$
   (a) Is $\oplus_1$ associative on $\mathcal{M}$? Prove or find a counter example.
   (b) Does $\oplus_1$ have an identity on $\mathcal{M}$? (Of course this involves a (possible short) proof).
   (c) Is $\langle \mathcal{M}, \oplus_1 \rangle$ a group? Either prove it is or show that one of the axioms fail.

3. Define a binary operation $\oplus_2$ on $\mathcal{M}$ by:
$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \oplus_2 \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 \\ c_2 & d_1 + d_2 \end{bmatrix}.$$
   (a) Is $\oplus_2$ associative on $\mathcal{M}$? Prove or find a counter example.
   (b) Does $\oplus_2$ have an identity on $\mathcal{M}$?
   (c) Is $\langle \mathcal{M}, \oplus_2 \rangle$ a group? Either prove it is or show that one of the axioms fail.

4. Define a binary operation $\oplus_3$ on $\mathcal{M}$ by:
$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \oplus_3 \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + d_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}.$$
   (a) Is $\oplus_3$ associative on $\mathcal{M}$? Prove or find a counter example.
   (b) Does $\oplus_3$ have an identity on $\mathcal{M}$?
   (c) Is $\langle \mathcal{M}, \oplus_3 \rangle$ a group? Either prove it is or show that one of the axioms fail.

5. Let $\mathcal{M}' = \left\{ \begin{bmatrix} a & 2 \\ -1 & d \end{bmatrix} : a, d \in \mathbb{Z} \right\}$ be a subset of $\mathcal{M}$. Show that $\langle \mathcal{M}', \oplus_2 \rangle$ is a group (Hint: Do you need to show that $\oplus_2$ is associative on $\mathcal{M}'$ given that you already showed in for the larger set $\mathcal{M}$?).