Additional Group Problems Assignment 25

1. Show the quaternion group Q_8 is not isomorphic to D_4 and thus there are at least two (and in fact only two) non-abelian groups of order 8.

2. Let R be a ring let $M_2(R) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a, b, c, d \in R \right\}$.

   (a) Show the distributive law holds on $M_2(R)$ (and hence it is a ring since the other axioms are easy to see given what we have done already). That is show that if $A, B, C \in M_2(R)$ then $A(B + C) = AB + AC$.

   (b) Show that if R is a ring with unity then so is $M_2(R)$.

3. Consider the field $\mathbb{Z}_p$, for a prime p. For $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_2(\mathbb{Z}_p)$ let $\det(A) = a_1 a_4 - a_2 a_3$ (calculated in $\mathbb{Z}_p$ of course).

   (a) Suppose A is such that $\det(A) \neq 0$, show that

   $$A^{-1} = (\det(A)^{-1}) \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$$

   is the multiplicative inverse of A in $M_2(\mathbb{Z}_p)$. (Note: $\det(A)^{-1}$ is the multiplicative inverse of $\det(A)$ in $\mathbb{Z}_p$ and $-a_2$ is the additive inverse in $\mathbb{Z}_p$.

   (b) Let $A = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$. Find the inverse of A where A is an element of:

      i. $M_2(\mathbb{Z}_7)$

      ii. $M_2(\mathbb{Z}_{11})$

4. Suppose R is a ring with unity. Show $U(R)$, the set of units in R (the is elements with a multiplicative inverses) forms a group under the multiplication of R.