Additional Problem Assignment 1

1. Remember that if a, b are integers (i.e. if $a, b \in \mathbb{Z}$) then a is said to divide b (written $a|b$) if there exists $k \in \mathbb{Z}$ such that $b = a \cdot k$.

   (a) Show that $a|a$ for all $a \in \mathbb{Z}$.
   PROOF: Since $a = a \cdot 1$ and $1 \in \mathbb{Z}$ it follows by the definition of $|$ that $a|a$.

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   Note: a can be 0 since $0|0$ is a true statement. Thus you should not divide by a. As you can see all of these problems can be done without any division.

   (b) Show that if $a, b, c \in \mathbb{Z}$ and $a|b$ then $a|bc$. PROOF: Since $a|b$ there exists $k \in \mathbb{Z}$ such that $b = a \cdot k$. Thus $cb = c(ak) = (ck)a$ by commutativity and associativity of multiplication of integers. Since $ck \in \mathbb{Z}$, if follows that $a|cb$.

   $\triangle$

   (c) Show that if $a, b, c \in \mathbb{Z}$ and if $a|b$ and $b|c$ then $a|c$. PROOF: Suppose $a|b$ and $b|c$ there exists $k_1, k_2 \in \mathbb{Z}$ such that $b = k_1 a$ and $c = k_2 b$, it follows that $c = k_2 k_1 a$ and since $k_1 k_2 \in \mathbb{Z}$ if follows that $a|c$.

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   Note: Make sure you choose different symbols for k_1 and k_2 since they could be different (you could use k and l or any other two variables). Some people called both k which causes problems.

   (d) Show that if $a, b, d, r, s \in \mathbb{Z}$ and if $d|a$ and $d|b$ then $d|ar + bs$. PROOF: Suppose that $d|a$ and $d|b$ then there exists $k_1, k_2 \in \mathbb{Z}$ such that $a = k_1 d$ and $b = k_2 d$. Thus $ar + bs = k_1 dr + k_2 ds = d(k_1 r + k_2 s)$ by the distributive law of addition over multiplication. Since $k_1 r + k_2 s \in \mathbb{Z}$ it follows that $d|ar + bs$.

   $\triangle$

   Note: Notice in all cases, I started with the hypothesis and moved to the conclusion. It is okay to include what is to be shown but make sure to clearly state "we want to show" or wts for short. However, make sure to never use the conclusion, it is only mentioned as a road sign in the proof. Also make sure you understand that $a|b$ is a statement and thus is very different from the rational number $\frac{b}{a}$ which is indeed an integer if the statement is true (and if $a \neq 0$). Notice it does not make sense to add statements, you can only add numbers.

   When ever a variable is introduced you must put in in context. This can be done with a quantifier, "for some $k \in \mathbb{Z}$" or "for all $k \in \mathbb{Z}$" or it can be done by defining the variable in terms of previously defined quantities like "let $k = ab + c$" or "where $k = ab + c$". You should not start using variables that have not been introduced in one of these ways unless they are in the original problem.

2. Suppose A, B, C are sets. Give the definition of what it means that $A \subseteq B$ that is A is a subset of B. **Answer:** For all x, if $x \in A$ then $x \in B$.

3. Show by mathematical induction for all $n \in \mathbb{N}$, $2^n > n$. Remember that $N = \{1, 2, 3, \ldots\}$ is the set of natural numbers.

PROOF: We will prove this by mathematical induction. For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $2^n > n$. We will prove that $P(1)$ is true and if $P(k)$ holds for any $k \in \mathbb{N}$ then $P(k+1)$ is also true.

Base Case: Notice that $P(1)$ is the statement $2^1 > 1$ which clearly holds. Thus $P(1)$ is true.

Inductive Step: We will assume that if for some k, $P(k)$ is true (this is called the inductive hypothesis). That is we will assume that for this k, $2^k > k$. Now we will show that his implies that $P(k+1)$ is true. Observe that:

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&> 2k \quad \text{by the inductive hypothesis} \\
&= k + k \\
&\geq k + 1 \quad \text{since } k \geq 1.
\end{aligned}$$

Hence $2^k + 1 > k + 1$. This is exactly $P(k+1)$. Hence $P(k)$ implies $P(k+1)$.

Thus by the principle of mathematical induction $P(n)$ holds for all $n \in \mathbb{N}$. Thus $2^n > n$ for all $n \in \mathbb{N}$.

△

Note: Notice the inductive hypothesis is that the statement is true for some element of the natural numbers not that it is true for all natural numbers. Indeed the latter is what you are trying to show. Also the proof should end with a conclution of what has been shown. I think it is confussing to end with the inductive step, better to say "thus by the principle of mathematical induction . . . ".