Exam 2 Review Questions

These questions are designed for you to do to prepare for the exam. Since you are not turning them in the point is not to finish the problems but to really understand them. Of course this is always the point of homework! Don't fool yourself in believing you "kind of" get it.

1. Determine if the following are true or false. Provide either a proof or a counterexample.
   (a) If $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ are topological spaces then the collection:
   $$\{U \times V : U \in \mathscr{T}, V \in \mathscr{S}\}$$
   is a topology on $X \times Y$.
   (b) If $f : X \to Y$ is continuous and $A \subseteq X$ is connected then $f(A)$ is connected.
   (c) If $f : X \to Y$ is continuous and $A \subseteq X$ is disconnected then $f(A)$ is disconnected.
   (d) If $f : X \to Y$ is an open bijection and $B \subseteq Y$ is connected then $f^{-1}(B)$ is connected.
   (e) If $f : [0, 1] \to [0, 2]$ is continuous (in $\mathscr{U}$) then f has a fixed point.
   (f) If $f : [0, 2] \to [0, 1]$ is continuous (in $\mathscr{U}$) then f has a fixed point.
   (g) If $(X, \mathscr{T})$ has the fixed point property then so do all subspaces of X.

2. Give 3 different bases for the topology $\mathscr{U} \times \mathscr{H}$ on $\mathbb{R}^2$.

3. Find all connected subsets of $\mathbb{R}$ under the $\mathscr{H}$-topology. Prove your results, that is show all subsets you claim to be connected are connected and all subsets you claim are disconnected are disconnected.

4. Let $(X, \mathscr{T})$ be a topological space. We say that a nonempty $A \subseteq X$ is a connected component of X if A is open, closed and connected.
   (a) Find the connected components of $X = (0, 1] \cup (4, 5) \cup \{9\}$ with the usual topology on X.
   (b) Prove that if A is a connected component of X and $A \subseteq B \subseteq X$ then B is connected if and only if $A = B$.
   (c) Show that if $h : X \to Y$ is a homeomorphism and $A \subseteq X$ is a connected component of X then $h(A)$ is a connected component of Y.

5. Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be topological spaces and $y_0 \in Y$. Show that X is homeomorphic to $X \times \{y_0\}$ (under the product topology).

6. Draw the following subsets of $\mathbb{R}^2$ and determine if they are connected, by either proving they are or drawing two open subsets that disconnect them:
   (a) $([0, 2] \times [0, 2]) \cup ([3, 4] \times [1, 4])$
   (b) $([0, 2] \times [0, 2]) \cup ([1, 3] \times [1, 4])$
   (c) $([0, 2] \times [0, 2]) \cup ([2, 3] \times [2, 4])$
   (d) $([0, 2] \times [0, 2)) \cup ([2, 3] \times (2, 4])$