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1 Introduction

My research interests focus mainly on 3-manifolds, hyperbolic geometry, and flows on manifolds. Thurston's Geometrization Conjecture maintains that every compact 3-manifold can be decomposed into geometric pieces, each piece admitting one of 8 possible geometries; this conjecture has already been proven in the case that the 3-manifold is Haken (contains a closed incompressible surface). In many ways 3-manifolds admitting hyperbolic geometry are at the same time the most common and the least well understood of the 8 geometries; for this reason my main interest is in hyperbolic manifolds.

Many approaches have been developed to try to understand 3-manifolds. For example, one might study which "meaningful" flows a 3-manifold will admit, in hopes that the right definition of meaningful might lend some insight into the topology or geometry of the 3-manifold. One useful definition of meaningful is "quasigeodesic:" a flow is quasigeodesic if each flow line is efficient at measuring distances up to a bounded distortion. Until recently, the only known examples of quasigeodesic flows were on closed hyperbolic 3-manifolds; my Ph.D. thesis provided the first known examples of quasigeodesic flows on cusped (non-closed) hyperbolic 3-manifolds. Building on this result, I propose to

> **Project 1:** Discover which Dehn fillings of hyperbolic 3-manifolds which fiber over the circle yield manifolds for which the quasigeodesic flow on the cusped manifold extends to a quasigeodesic flow on the filled manifold.

The study of flows is part of a larger subject of foliations and laminations of 3-manifolds. Thurston associates a lamination to each geometrically infinite end of an infinite volume manifold with boundary; the ending lamination conjecture states, essentially, that these laminations suffice to determine the manifold up to quasi-isometry.

At a talk I gave on my thesis problem, Lee Mosher suggested that the techniques I developed in my thesis might be successfully applied to this conjecture, in the case of a punctured surface group in which there is a positive lower bound on injectivity radius everywhere except in the cusps. I sent this idea to Yair Minsky, who has contributed a great deal to this conjecture, and he said it would be worth a try. Therefore, I propose to

> **Project 2:** Develop the techniques in my thesis towards Thurston's ending lamination conjecture in the special case of a punctured surface group in which there is a positive lower bound on injectivity radius everywhere except in the cusps.

Yet another approach to understanding 3-manifolds is to ask which incompressible surfaces a 3-manifold M will admit. In the case that M is a knot complement, a properly embedded incompressible surface with boundary describes a curve on the torus boundary of M called the boundary slope. I have written a computer program to calculate a list of boundary slopes for any knot complement, based on a technique developed by Culler and Shalen. This computer program finds many valid boundary slopes, but it does not detect as many as it could. Therefore, in the coming months I will

> **Project 3:** Expand my computer program, which computes boundary slopes of knot complements, to find even more boundary slopes, by using a more subtle implementation of the technique used in the current version.

I describe each of these problems in greater detail below.

2 Quasigeodesic Flows

The main focus of my research has been in the study of flows on 3-manifolds. A flow on a manifold is a foliation of the manifold by coherently oriented one-manifolds. Flows are common; in fact, every closed 3-manifold has a flow given by the integral curves of a continuous, nowhere zero vector field, as the Euler characteristic of such a manifold is 0.

For a flow on a manifold to be useful in studying a 3-manifold, it must be connected to the topology or geometry of the manifold. For example, you might ask which 3-manifolds will admit a flow where each flow line is geodesic. In the case that the 3-manifold is closed and hyperbolic, the answer is "none": Zeghib showed that a closed hyperbolic 3-manifold will not admit any flow where each flow line is geodesic [Z].

If a manifold M admits a flow where all flow lines are geodesic, and the metric on M is changed slightly, then these flow lines are merely quasigeodesic in the new metric. A path is quasigeodesic in a given metric if in universal cover, the length of a segment of the lifted path between two points is a bounded multiple of the distance between the endpoints of the segment. The concept of quasigeodesic has the advantage of being independent of the metric in the manifold.

Perhaps a more natural question to ask, then, is whether a given manifold will admit a quasigeodesic flow. A flow is (uniformly) quasigeodesic if all sufficiently long segments of flow lines are quasigeodesic with a uniform quasigeodesic constant.

My main area of research is quasigeodesic flows, ones in which each flow line is efficient at measuring distances in the manifold up to a uniform scalar multiple. Two natural questions arise: which 3-manifolds admit quasigeodesic flows, and what properties of the 3-maifold can you ascertain once you know a 3-manifold will admit a quasigeodesic flow. My research has been focused on the first of these questions.

2.1 Motivation

The wealth of information one can obtain from a quasigeodesic flow on a 3-manifold is only beginning to be understood. A few examples are outlined below.

Of the 8 geometries described in [S], manifolds admitting a hyperbolic geometry are the most studied as they are both the most common and the least well understood of the 8 geometries. Finding a quasigeodesic in a hyperbolic 3-manifold is essentially as good as finding a true geodesic, for a quasigeodesic in the universal cover $\mathbf{H}^3$ is always a bounded distance from some minimal geodesic [Th1]. This, among other reasons, makes quasigeodesics extremely useful in studying hyperbolic manifolds [Th1] [Th3] [Mor] [Ca].

Mosher showed that quasigeodesic pseudo-Anosov flows on hyperbolic manifolds can be used to compute the Thurston norm [Mo1][Mo2]. Every rational homology class of a 3-manifold M can be represented by a surface; the Thurston norm is a norm on these surfaces. The unit ball of the Thurston norm is a polyhedron in which the fibers of a fibration of the manifold form the interior of a union of the top dimensional faces. Thus all fibrations of M (indeed, all incompressible surfaces in M) are represented as points on a face of the unit ball, and if one finds one point of a face of the unit ball corresponding to the fiber of a fibration of the 3-manifold, then the surfaces represented by all points in that face correspond to fibers of a fibration of the manifold.

Also, Fenley has shown that there is a large class of closed hyperbolic 3-manifolds admitting codimension one foliations with good large scale geometric properties, in that leaves reflect very well the geometry in the large of the universal cover and are geometrically tight. The foliations in Fenley's result are the singular stable and unstable foliations of a pseudo-Anosov flow; his proof makes use of quasigeodesic property of this flow.

2.2 History and Progress

Thurston [Th2] has shown that if closed orientable 3-manifold which fibers over the circle is hyperbolic, then it can be represented as a product of a hyperbolic surface (the fiber, F) with the unit interval $[0,1]$, with $F \times \{1\}$ glued to $F \times \{0\}$ using a pseudo-Anosov monodromy map. The manifold M is then called the suspension of F. Thus M is covered by $F \times \mathbf{R}^1$; the flow on M obtained by projecting the 1-manifolds $\{x\} \times \mathbf{R}^1$ from $F \times \mathbf{R}^1$ to M is called the suspension flow.

In their seminal work in the 1970's, Cannon and Thurston [CT] showed that the suspension flow on a closed hyperbolic 3-manifold M which fibers over the circle is quasigeodesic. In this work, Cannon and Thurston used the pseudo-Anosov property of this monodromy map to construct a metric on M called the singular Solv metric, in which the flow lines are easily seen to be geodesic. As all metrics on a closed manifold are quasi-isometric (by compactness), it follows that the flow lines are quasi-geodesic in the hyperbolic metric on M. Later, Zeghib [Z] gave an elementary proof based on compactness that if M is any closed (not necessarily hyperbolic) 3-manifold fibering over the circle, then any flow transverse to the fiber is quasigeodesic.

Combined works of Gabai, Mosher, and Fenley show that any closed, oriented, hyperbolic three-manifold with nontrivial second homology admits many quasigeodesic flows, as follows: Gabai [Ga] uses $H_2(M) \neq 0$ to construct a Reebless, finite depth 2-dimensional foliation on M. Mosher [Mo3] produces a pseudo-Anosov flow Φ which is "almost" transverse to these finite depth foliations, and joint work of Fenley and Mosher [FM] shows that the lifted flow $\widetilde{\Phi}$ on $\mathbf{H}^3$ has quasigeodesic flow lines.

In each of the above cases, the manifold under consideration is closed, and the flow described is transverse to a fiber of a fibration or to a finite depth foliation. In general it is difficult to control the behavior of a flow unless one has such conditions; in fact, the question of whether the suspension foliation on a similar cusped manifold is quasigeodesic has remained open for 20 years.

If a hyperbolic 3-manifold fibering over the circle has cusps, then Zeghib's compactness argument does not apply. The Cannon-Thurston technique breaks down as well, for the singular Solv metric is not quasi-isometric to the hyperbolic metric in the cusped case as it places the cusps at only a finite distance in the manifold. A priori, it seems possible that the flow might have a sequence of flow lines, each entering the cusp more and more deeply and turning back more and more sharply, so that although each flow line is a quasigeodesic, there is no upper bound to this sequence of quasigeodesic constants. It is this infinite part of the manifold that causes all the problems.

In my Ph.D. thesis, I answered the question in the case where M is not compact, but instead has finitely many cusps.

Theorem [Hoffoss] *A cusped hyperbolic 3-manifold which fibers over the circle admits a flow (the suspension flow) which can be isotoped to be uniformly quasigeodesic.*

The proof follows the spirit of Cannon and Thurston's proof in the closed case: a metric is constructed in which it is possible to show that the flow lines are modifying the singular Solv metric in neighborhoods of the cusps to push the cusps out to infinity produces a metric which is quasi-isometric to the hyperbolic metric on the manifold. I showed that the suspension flow is uniformly quasigeodesic in this modified metric, and thus it is are quasigeodesic (with perhaps a different quasigeodesic constant) in the hyperbolic metric on M. However, unlike the Cannon-Thurston case, it is not at all obvious that the flow lines are geodesic or even quasigeodesic in this modified singular Solv metric; instead, a fairly complicated argument was needed to establish this result.

2.3 My Goal

The manifolds I studied in my thesis have torus boundary components. In light of my result, a natural question to ask is: when one can extend the quasigeodesic flow of such a cusped 3-manifold to a quasigeodesic flow on a Dehn filling of the manifold?

Dehn surgery on closed orbits was introduced by Goodman in 1983. Given an oriented Anosov flow on a 3-manifold M, i.e. one whose (weak) stable and unstable foliations are oriented, then for every closed orbit γ of the flow there is a preferred basis for Dehn surgery space on γ: the meridian $(0,1)$ is the trivial surgery, and the longitude $(1,0)$ is surgery parallel to the weak stable foliation. Goodman proved that Dehn surgery with coordinates $(n,1)$ on γ yields an Anosov flow on the surgered manifold.

Although Fenley [Fe1] showed that the Anosov flows Goodman creates are *not* quasigeodesic, I would like to discover for which Dehn fillings the quasigeodesic flow in the cusped manifold extends to a quasigeodesic flow in the filled manifold. I believe an analogy of the techniques in my thesis might help me to find the answer.

3 Applying Thesis Techniques to the Ending Lamination Conjecture

Yet another way one might approach understanding 3-manifolds is to study their "end behavior." An infinite volume 3-manifold with boundary composed surfaces of genus at least 2 has boundary components $S \times [0, \infty)$ (where S is a boundary surface); these are called the ends of the manifold. Ends come in two different flavors: geometrically finite ends, which are fairly well understood, and geometrically infinite ends, which are not as well understood.

Bonahon and Thurston [Bo][Th1] showed that a geometrically infinite end with no parabolics has a sequence of simple closed curves with geodesic representations that exit the end, and that this sequence converges to a unique lamination on the surface S. This lamination is called the ending lamination associated to this geometrically infinite end. Thurston's ending lamination conjecture states that the ending laminations on the geometrically infinite ends of a manifold, together with the conformal boundaries of its geometrically finite ends (and the homeomorphism type of the manifold), suffice to determine the manifold up to quasi-isometry.

3.1 History and Progress

In 1993, Minsky proved Thurston's ending lamination conjecture for the special case of hyperbolic manifolds (with incompressible boundary) admitting a positive lower bound on injectivity radius [Mi1]; in May, 1997 he established the ending lamination conjecture for the punctured torus [Mi2].

3.2 My Goal

In August 1997, Mosher suggested to me that the techniques I developed in my thesis might successfully be used to address Thurston's Ending Lamination Conjecture in the case of a punctured surface group in which there is a positive lower bound on injectivity radius everywhere except in the cusps. I told Minsky of Mosher's idea; he responded:

> I imagine in that case that your techniques to get the generalization of the Cannon-Thurston map, together with some of the ideas in the work I did on manifolds with a positive lower bound on injectivity radius -everywhere- (Topology 93 and JAMS 94), might give the conjecture. That would certainly be worth attempting.

I am not an expert in this area, and I understand that the ending lamination conjecture is a very difficult problem which is being worked on my many mathematicians. However, I am interested in the problem, and I intend to try Mosher's suggestion. I will not be isolated in my pursuit of this question: Rice colleague Richard Evans (student of Dick Canary) has studied the ending lamination conjecture for some time and has offered to discuss the problem with me, and Mike Wolf will also be a good resource. While my techniques may not prove the ending lamination conjecture even in this restricted case, I hope that development of my techniques in this direction might add some insight to the problem.

4 Computing Boundary Slopes

A popular and productive approach to understanding 3-manifolds is to investigate which incompressible surfaces live inside the manifold. If M is a 3-manifold with a single torus boundary component (for example, a knot complement), one can ask which incompressible, boundary incompressible surfaces F with boundary live inside M The boundary of each such incompressible surface is a collection of parallel (p, q)-curves on the torus boundary of M, wrapping p times in the meridinal direction and q times a longitudinal direction the torus. The fraction p/q is referred to as a slope on the boundary of the torus, or a boundary slope. Each 3-manifold M with a torus boundary component has only finitely many boundary slopes; the set of boundary slopes of M is an invariant of the manifold.

4.1 History and Progress

Culler and Shalen [CS] introduced a new method of constructing incompressible surfaces by using representations of fundamental groups of hyperbolic 3-manifolds into $\mathrm{SL}_2(\mathbf{C})$. They showed that each ideal point of an algebraic curve in the character variety corresponds to an incompressible and boundary incompressible surface in the hyperbolic 3-manifold. Corresponding to each such ideal point is a sequence of representations limiting on the ideal point. This gives rise to a degeneration of the ideal triangulation of the 3-manifold; the ideal tetrahedra degenerate to ideal triangles which, glued together, construct the incompressible surfaces.

Based on this idea, I wrote a computer program in 1993 to compute boundary slopes of a manifold with a torus boundary component. This program takes as input the data for an ideal triangulation of the manifold from SnapPea (a widely used topology and geometry computer program). If there are n tetrahedra in such an ideal triangulation, then there are 3^n possible ways to degenerate each tetrahedra to an ideal triangle. My program attempts all possible ways for the ideal tetrahedra to degenerate while still satisfying their glueing equations. For each such successful degeneration, my program outputs the corresponding boundary slope. However, some enhancements to this computer program are needed before it is ready to be released.

Considerable interest has already been shown in my completing this work. Jeff Weeks (author of SnapPea) has asked to include my program into SnapPea when I produce the final version. Also, two mathematicians (Suho Park from Korea and Paul Libbrecht from Germany) have independently contacted me about my computer program, asking for a copy once I have finished it.

4.2 My Goal

This computer program detects many boundary slopes; I believe employing a more subtle algorithm in my computer program will allow me to detect even more slopes. During the coming year, I will complete this enhancement to my computer program.

Montesinos has compiled a partial list of boundary slopes for knot complements up to 10 crossings. My program has found a boundary slope which does not appear on Montesinos's list. Although this could mean that I have found a boundary slope which was previously unknown, there are indeed cases where tetrahedra degenerate but representation does not blow up; in other words, this may not be a true boundary slope. Therefore, I would like to fortify my computer program to test whether slopes arising from these degenerations do indeed correspond to a representation which blows up, making the output from my program robust.

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