Lectures – Math 128 – Geometry – Spring 2002

$\sim\sim\sim$ Day 1 $\sim\sim\sim$

Introduction

1. Introduce Self
2. Prerequisites – stated prereq is math 52, but might be difficult without math 40

Syllabus Go over syllabus

Overview of class

- Recent flurry of mathematical work trying to discover shape of space
- Challenge assumption space is flat, infinite
- space picture with Einstein quote
- will discuss possible shapes for 2D and 3D spaces
- this is topology, but will learn there is intrinsic link between top and geometry
- to discover poss shapes, need to talk about poss geometries
- Geometry = set + group of transformations
- We will discuss geometries, symmetry groups
- Quotient or identification geometries give different manifolds / orbifolds
- At end, we'll come back to discussing theories for shape of universe

How would you try to discover shape of space you're living in?

2 Dimensional Spaces – A Square

1. give face, 3D person can do surgery
2. red thread, possible shapes, veered?
3. blue thread, never crossed, possible shapes?
4. what about NE direction?

List of possibilities: classification (closed)

1. list them
2. coffee cup vs donut

How to tell from inside – view inside small torus

1. old bi-plane game, now spaceship
2. view in each direction (from flat torus point of view)
3. tiling pictures
4. fundamental domain – quotient geometry
5. length spectra can tell spaces apart
6. finite area
7. which one is really you?
8. glueing, animation of folding torus
9. representation, with arrows
10. discuss transformations
11. torus tic-tac-toe, chess on Friday

Different geometries (can shorten or lengthen this part)

1. describe each of 3 geometries
2. angle sum
3. gauss

3 Dimensional Spaces – maybe not infinite

1. other possibilities, maybe not infinite; hard to imagine from inside
2. 3 torus
   (a) glue room – what do you see
   (b) ghost images, mirrored room
   (c) play catch with self
   (d) which one is you?

(e) Einstein Picture

(f) glue with twist, etc

3. snappea picture

4. There is no complete list!!

5. 8 possible geometries

6. Vast, vast majority are hyperbolic

7. what topologists are trying to do

8. snap pea length spectra

We don't see this, does that mean finite is false?

1. large

2. time for light to travel, see things as they were billions of years ago

3. conjecture: finite and hyperbolic

Homework:

1. Buy text

2. Read Chapters 1, 2, 3 – don't look at hints to exercises

3. We'll discuss chapter 3, and do exercises Wednesday

~~~~~~~~~ Day 2 ~~~~~~~~~

Chapter 3 – Vocabulary

Topology vs. Geometry

Example Deforming a surface, same top, different geom

Definition: The *topology* of a surface (or 3-dim space consists of the aspects of the nature of the surface that do not change when you deform the space. Two spaces have the same topology if one can be deformed into the other, without making any tears. For (closed, orientable) surfaces, topology essentially boils down to how many holes you have.

Definition The *geometry* of the surface consists of those properties that do change when the surface is deformed. Curvature, distances, angles, areas.

Note Flat torus and doughnut torus have same topology, different geometry. We'll talk about flat torus when we're talking about geometry.

Experts Actually, a *topology* on a set is a declaration of which subsets of that set will be called "open". Two sets are the same topologically if there is a map from one set to the other that sends open sets in one topology to open sets in the other. As extra credit, look up this definition, and show that a line segment with endpoints identified has the same topology as a circle.

Intrinsic vs. Extrinsic

Example, topology : Annulus, annulus with double twist.

Definition: Two surfaces have same *intrinsic* topology if they look the same from inside the space.

Definition: Two surfaces have same *extrinsic* topology if one can be deformed within 3d space to look like other – if they can be made to look the same from outside the space.
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Example, geometry: Bending a sheet of paper. Same intrinsic, different extrinsic.

Comment: We have extrinsic view of surfaces, so easier to understand. Some of difficulty in finding out shape of our space is that we can only perceive things intrinsically.

Comment: Do not have to think of our space as living in some other, higher dimensional space. It can exist on its own.

Local vs. Global

Definition: Local properties can observe within small region, global properties require thinking of space as a whole.

Examples:

- Measure angles from 3 mountaintops and get sum ¡ 180 (local)
- As civilization spreads, Flatlanders discover area of their space to be finite. (global)

Local geometry and Global Topology (customary pairings)

1. Name 2 spaces with same local geom, different global top (flat torus and plane)
2. Name 2 spaces with same global top, different local geom (flat torus and donut surface)

Definition: A *2-dimensional manifold* is a space with the local topology of a plane. A *3-dimensional manifold* is a space with the local topology of a ordinary 3-space.

Homogeneous vs. Nonhomogeneous Geometries:

Definition: A *homogeneous* manifold is one whose local geometry is the same at all points. A *nonhomogeneous* manifold is one whose local geometry varies from point to point.

Closed vs. Open

Definition: Intuitively, *closed* means finite and *open* means infinite. But not exactly. Closed means “a finite distance across”, and open means “an infinite distance across.”

Examples

1. circle
2. line
3. two holed donut surface
4. sphere
5. plane
6. infinitely long cylinder
7. flat torus

Note Book mainly talks about closed manifolds.

Manifold, Manifold-with-boundary, and cusp

Definition A *manifold with boundary* is one that has an edge.

Note: In this book both closed and open imply manifold has no boundary.

Example: torus with cusp

Examples: Which have boundary

1. circle
2. line
3. line segment
4. two holed donut surface
5. sphere
6. hemisphere
7. plane
8. infinitely long cylinder
9. flat torus

$\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim$ Day 3 $\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim$

Torus and Klein bottle games in the computer lab

~~~~~~~~~~ Day 4 ~~~~~~~~~~

Complex Numbers

****More in-class examples needed, especially of multiplying a complex number by its conjugate, say $(x+1)+i(y^2-4)$ by its conjugate. Also to see that mod z squared is z times z conjugate

Useful way to study analytic geometry in the plane

Definitions:

- A *complex number* is a point $z = (x, y)$ in the Cartesian plane. The x axis is the "real" axis and the y axis is the "imaginary" axis.
- x value is real part of z: $x = \mathrm{Re}(z)$
- y value is imaginary part of z: $x = \mathrm{Im}(z)$
- Also write $z = x + iy$

Examples:

- Put points on plane
- Cover following ideas using algebra, geometry, examples:
  - Vector addition
  - Scalar multiplication
  - Modulus
  - Distance Between points
  - Conjugate

Multiplication of Complex Numbers

Show example using rectangular coordinates. For geometry, see later.

Polar Form of Complex Numbers

- Draw picture using polar coordinates
- Definition: $e^{i\theta} = \cos\theta + i\sin\theta$.
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- Show polar form of generic complex number z.
- Convert polar $\leftrightarrow$ rectangular coordinates

Definition: Let $z = re^{i\theta}$. The *argument of* $z = \arg(z) = \theta \pmod{2\pi}$

Geometry of Complex number multiplications

Now show multiplication of complex numbers using polar coordinates

Examples:

- Sketch $|4z + 2i| = .5$ (factor 4 out), or do long way
- Classwork: Sketch $\operatorname{Im}\left(\dfrac{z}{z-i}\right) < \dfrac{1}{2}$

Lines

Parametrize a line in the complex plane.

Homework:

- Chapter 2 #1d, 2b, 3b, 5, 6, 9, 10d, 11, 12, 14g, 18d,i,k, 19b

~~~~~~~~~ Day 5 ~~~~~~~~~

Geometric Transformations

Main Point

Geometries are classified in terms of their group of transformations – the maps from a set to itself that preserve the geometry. More precise definition to follow.

Definition

A *transformation* is a one-to-one function whose domain and range are same set. We will primarily be considering functions from the complex plane to itself.

Notes

- Describe one-to-one function
- Think of as motion of complex plane, moving each point to its image

Examples (derive formulas for these in class)

Translation

$w = Tz = z + b$, b a complex number

Rotation

- Rotation about origin $Tz = e^{i\theta}z$
- Example: Find formula for rotation by $45°$ about the point i.
- Mention the idea of conjugacy.
- Generic rotation around complex number v, as composition of 3 functions.

Reflection

- Special case: $Tz = \overline{z}$
- Find formula for reflection about arbitrary line through origin.
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- Classwork: Find formula for reflection about line $z = 2$

Homothety

$Tz = kz$, k a positive real number.

Inversion

- $Tz = \frac{1}{z}$ for $z \neq 0$.
- Prop: Inversion swaps inside of unit circle with outside

  Proof: Write z in polar form, and invert.
- Prop: Inversion swaps upper half plane with lower half plane

  Proof: Same as above.
- Prop: Inversion swaps origin with infinity
  - discuss point at infinity
  - complex plane plus point at infinity can be identified with sphere.
- Prop: Inversion takes
  - lines to circles passing through origin
  - circles to circles
  - circles through origin to straight lines

  Proof: in text
- Draw circles before and after.

Conformal maps

- Definition: A *conformal* map preserves angles.
- Which transformations above preserve angles?
  - Translation, rotation, reflection obvious (rigid motion)
  - Homothety
  - Inversion
- What is angle between two circles?

Homework

- Chapter 3 in handout #1-3, 5bc, 6, 7, 8

~~~~~~~~~~ Day 6 ~~~~~~~~~~

Stereographic Projection :

- From unit sphere onto complex plane, projection from North Pole
- Draw picture, complex plane goes through center of sphere!
- North pole goes to infinity
- Useful in studying spherical geometry
- Circle through north pole projects to line
- Stereographic projection is conformal

Formula for Stereographic Projection

- Parametric equation of line from North Pole to generic point (a, b, c) is
$$(x, y, z) = (0, 0, 1) + t(a, b, c - 1)$$
- Find t value where $z = 0$. Will get $t = \dfrac{1}{1-c}$,
$$S(a, b, c) = \frac{a}{1-c} + i\frac{b}{1-c}$$

Lifts of transformations

- We have a 1-1 correspondence between unit sphere and complex plane + ∞. Using this correspondence, we can *lift* (i.e. transfer) any transformation of one set into a transformation of the other.
- Example of lifting rotation about origin in complex plane to action on sphere.

Definition

A *covering transformation* from a set D to a set R is a continuous function S from D onto R. You say "D covers R via S".
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Definition

A transformation g (in the "covering" set) is a *lift* of a transformation f (in the "covered" set) if diagram commutes:

$$S(g(z)) = f(S(z)).$$

Draw both picture and commutative diagram

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{g} & \mathbb{S}^2 \\ \mathrm{S}\downarrow & & \downarrow\mathrm{S} \\ \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R}^2 \end{array}$$

Example

What is the lift of rotation about the origin in the complex plane to the unit sphere?

- write intuitive formula for lift
- show it satisfies commutative diagram

Example

What is the lift of inversion in the unit circle in the complex plane to the unit sphere?

- Take point (a, b, c) down to complex plane, then over by inversion

$$\begin{aligned} IS(a,b,c) &= I(\frac{a+bi}{1-c}) \\ &= \frac{1-c}{a+bi} \cdot \frac{a+bi}{a+bi} \\ &= \frac{(1-c)(a-bi)}{a^2+b^2} \\ &= \frac{(1-c)(a-bi)}{(1-c)(1+c)} \qquad \text{since } a^2+b^2+c^2=1 \\ &= \frac{a-bi}{1+c} \\ &= S(a,-b,-c) \end{aligned}$$

- Will recognize as image of $(a, -b, -c)$ under S, so $f(a, b, c) = (a, -b, -c)$.
- Lift is $180°$ rotation about the a axis.

Homework

- Chapter 3 #10, 11, 13, 15
- Read Chapter 4

~~~~~~~~~~ Day 7 ~~~~~~~~~~

The Erlanger Program

The EP is a way to describe / classify geometries. Often want to tell to what extent two geometries are same or different. The EP gives a way to decide whether a particular geometric idea belongs to a geometry or not.

Congruence

- One way to think: Two figures are *congruent* if they have identical geometrical properties, via measurement. For example, side lengths, angles, etc. So measurement implies congruence. This is the approach you used in high school
- In Erlanger Program, two figures are *congruent* if one can be moved so as to coincide with the other.
- So describing when two figures are congruent is same as describing which movements are allowed. In other words, want to specify set of functions to act as congruence transformations.
- Congruence must be an equivalence relation: Need properties of reflexivity, symmetry, transitivity.

Comment

A geometry, then, is a set plus its congruence transformations.

Definition

Let S be a nonempty set. A *transformation group* is a collection G of transformations $T : S \to S$ such that

1. G contains the identity
2. the transformations in G are invertible and their inverses are in G
3. G is closed under composition.

Discuss

Why do we need all these conditions on a transformation group?
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Definition

- A *geometry* is a pair (S, G) consisting of a nonempty set S and a transformation group G.
- S = *underlying space* of geometry.
- G = *transformation group* of geometry.

Definition

A *figure* is any subset A of underlying set S of a geometry (S, G).

Definition

Two figures A and B are *congruent* if there is a transformation T in G taking one to the other: $T(A) = B$.

Examples of Geometries

Euclidean Geometry

- (C, E), where C is complex plane and $E = \{T | Tz = e^{i\theta}z + b\}$.
- E is the set of rigid motions of the plane
- Have class show that (C, E) is a geometry.
- Give examples of figures which are congruent in this geometry, and examples of figures which are not congruent in this geometry.

Homework:

None

~~~~~~~~~~ Day 8 ~~~~~~~~~~

Background For Today's Talk

1. Goal: classify 3-manifolds
2. Recall for 2-manifolds – comes down to which loops don't suck down to points.
3. Lickorish - Wallace: Every closed, orientable, connected 3-manifold can be obtained by surgery on a knot or link in S^3
4. Explain S^3 as compactification of R^3
5. Explain knot
6. Explain removing a knot
7. Explain surgery
8.
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~~~~~~~~~ Day 9 ~~~~~~~~~

Recall Def of geometry

More Examples of Geometry

Translational Geometry

- (C, G) where $G = \{T|Tz = z + b\}$, set of all translations.
- Show (C, G) is a geometry.
- Give examples of figures which are congruent in this geometry, and examples of figures which are not congruent in this geometry.

Trivial Geometry

- (C, G) where $G = \{T|Tz = z\}$, trivial group.
- Show (C, G) is a geometry.
- Give examples of figures which are congruent in this geometry, and examples of figures which are not congruent in this geometry.

Invariants

Definition

Let (S, G) be a geometry. Let D be a set of figure in this geometry. Let T be a transformation from G.

1. The set D is *invariant* if, for every member B of D, $T(B)$ is also a member of D. This means, for each element B in D, D must contain all figures congruent to B.
2. A function f defined on D is *invariant* if

$$f(T(B)) = f(B).$$

   This means the function gives the same value for congruent figures.
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~~~~~~~~~ Day 10 ~~~~~~~~~

Recall Definition

Let (S, G) be a geometry. Let D be a set of figure in this geometry. Let T be a transformation from G.

1. The set D is *invariant* if, for every member B of D, $T(B)$ is also a member of D. This means, for each element B in D, D must contain all figures congruent to B.

2. A function f defined on D is *invariant* if

$$f(T(B)) = f(B).$$

   This means the function gives the same value for congruent figures.

Examples

- Triangles
  - D is set of all triangles
  - Euclidean geometry, D is invariant
  - Under what geometries is D not invariant?
- Area, Perimeter, and distance sum
  - these are functions on triangles.
  - area and perimeter are invariant under Euclidean geometry
  - sum of distances from vertices to origin is not invariant.
  - under which geometries is distance sum invariant?
- Lines parallel to x-axis: what geometries is this set invariant, not invariant?
- What are invariants under trivial geometry? (Every set and every function)

Erlanger Program says

Items appropriate to study about a geometry are its invariant sets and the invariant functions on those sets.

- Triangles belong to Euclidean geometry
- Area and perimeter of triangles belong to Euclidean geometry
- distance sum does not belong to Euclidean geometry
- Triangle, area, perimeter, direction or slope of a line belong to Translation geometry
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Useful Geometric Proof Technique resulting from Erlanger Program

- Want to prove something, call it W about some geometric figure F. (Need that something to be under Erlanger Program; in other words, need everything about W to be invariant.)
- To prove W for F, suffices to find a transformation T for which W is easy to prove for $T(F)$. For example, place in nice position in coordinate system.
- Example: Prove that the diagonals of a rectangle bisect each other (in Erlanger spirit)

Abstract Geometries and their Models

Definition

An *abstract geometry* is a collection of geometries which are "the same".

Example

Rotations of unit disk centered at origin about origin, Rotations of disk radius 3 centered at 5 are both geometries, but somehow fundamentally the same.

Definition

Two geometries (S_1, G_1) and (S_2, G_2) are *models* of the same *abstract geometry* if there is an invertible covering transformation $\mu : S_1 \to S_2$ so that every transformation in one geometry is a lift of a transformation in the other geometry. Can also say the geometries are *isomorphic* and μ is an *isomorphism*.

Commutative diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{T_1} & S_1 \\ \mu \updownarrow & & \updownarrow \mu \\ S_2 & \xrightarrow{T_2} & S_2 \end{array}$$

Example

Write geometries for D_1 and D_2, find isomorphism μ.

Note

Abstract geometry has various models; to study the geometry you study its models. Showing something in one model is equivalent to showing it in all models, and thus to showing it is true for the abstract geometry.

Homework

Chapter 4 # 1-4, 8-12, 14-17

~~~~~~~~~ Day 11 ~~~~~~~~~

Quotient Geometries

Definition of $< T >$

Let T be a transformation. Then the *group generated by the transformation* T is given by

$$< T >= \{T^n(z) | n \in \mathbb{Z}\} = \{I, T, T \circ T, T \circ T \circ T, ...T^{-1}, T^{-1} \circ T^{-1}, ...\}$$

Examples

1. $Tz = e^{\frac{\pi}{2}i}z$
2. $Tz = e^{2i}z$
3. $Tz = z + 3$

Subgroup

Let G and H be groups of transformations. Then H is a *subgroup* of G if H is a subset of G.

Examples

- Trivial subgroup, G itself
- G = Euclidean transformations, H = rotations about origin.
- If T is a transformation in a transformation group G, then $< T >$ is a subgroup of G.

General Group Theory – Quotient Groups

Group Requirements

In order to have a group, you need 2 things

1. A set of elements
2. A binary operation; a way to combine 2 of those elements to get another element.
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In our examples, our group elements are transformations, and our operation is composition.

Quotient Groups

Let G be a group of transformations, and let H be a subgroup of G. Then the quotient group of transformations G/H (sometimes read as "G mod H") is given as follows:

Slogan

The subgroup H counts as the identity element.

Equivalent transformations

Usually, two transformations T_1 and T_2 are equal if $T_1 \cdot T_2^{-1} = I$. Now, since H counts as the identity, let two transformations be "equal" (or equivalent) if

$$T_1 \cdot T_2^{-1} \in H.$$

Elements of quotient group

Collect together all the transformations that are "equal". Like partitioning a bag of M&M's. A pile of equivalent transformations constitutes an element of the quotient group.

Operation

Use the same operation as from G. This operation will take one whole pile to another whole pile.

Example

Let G be the Euclidean transformations:

$$G = \{T : Tz = e^{i\theta}z + b\},$$

and let H be translations:

$$H = \{T : Tz = z + a\}.$$

Then what is G/H?

- Esoteric way: Think of H as the identity. In other words, turn all translations into the identity. What's left?
- Elements are piles of equivalent transformations: Choose $T_1 z = e^{i\theta}z + b$ and $T_2 z = e^{i\phi}z + c$ in G. T_1 is equiv to T_2 if $T_1 \circ T_2^{-1} \in H$. Work this out.

~~~~~~~~~ Day 12 ~~~~~~~~~

Quotient Geometries

Example

Let G be the Euclidean transformations:

$$G = \{T : Tz = e^{i\theta}z + b, 0 \leq \theta < 2\pi\},$$

and let H be translations:

$$H = \{T : Tz = z + a\}.$$

Then what is G/H?

- Esoteric way: Think of H as the identity. In other words, turn all translations into the identity. What's left?
- Elements are piles of equivalent transformations: Choose $T_1 z = e^{i\theta}z + b$ and $T_2 z = e^{i\phi}z + c$ in G.
  - T_1 is equiv to T_2 if $T_1 \circ T_2^{-1} \in H$. Interpret.
  - What are the elements of G/H?
  - Compose up 2 elements of G/H

Example

- Let $(\mathbb{C}, G)$ be Euclidean geometry; i.e., $G = \{T : Tz = e^{i\theta} \cdot z + b\}$.
- Define $Tz = z + i$ (note $T \in G$). What points in $\mathbb{C}$ are congruent / equivalent under T?
- Consider the space you get by identifying / gluing together all points that are congruent / equivalent. What is result?
- Point out fundamental domain, covering space, gluing notation.
- In glued space, point out that T acts as the identity. In fact, $< T >$ acts as identity.

Definition

Let (S, G) be a geometry, and let T be a transformation in this geometry. Then the *quotient geometry* is the pair (S', G'), where

S' is obtained by gluing together all points congruent in S via T, and

G' is the quotient group $G/ < T >$.

We'll write this as $(S, G)/ < T >$ for shorthand.
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Examples

- $Tz = z + (1 + i)$
- $Tz = e^{\frac{\pi}{2}i}$
- Flat Torus – give transformations to mod out by, show which points are identified

Homework

Supplementary Homework sheet #1

~~~~~~~~~ Day 13 ~~~~~~~~~

Admin

Change exam 1 to March 20 (wed) from March 18 (mon after midterm holiday)?

Chapter 4 – Orientability

Hand out plastic strips

1. Is connecting together with no twists (*annulus*) same as connecting with 2 twists?
2. Is connecting with no twists same as connecting with 1 twist?
3. What is fundamental difference? Come back as mirror image
4. Careful, Mobius strip should not be thought of as 2-sided – do not come back on other "side" of strip
5. What happens when you go around strip twice?

Definitions

1. If, when you follow along a closed path, you come back mirror-reversed, this path is called an *orientation reversing* path.
2. Manifolds which do not contain orientation reversing paths are called *orientable*, manifolds that do contain orientation reversing paths are called *non-orientable*.

What Surfaces are Non-Orientable?

Mobius band

Just discussed

Klein Bottle

1. Draw flat Klein bottle
2. Draw several mobius bands on KB
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3. tic tac toe on klein bottle – figure 4.7 photocopy *****
4. Show how to "unwrap" transformations to get from fund domain to tiled picture

Geometry of Klein bottle

1. Ask about intrinsic geometry of Klein bottle
2. Book emphasizes geometry matches up perfectly, explain it's because it's a quotient of Euclidean geometry
3. If flat torus made of rubber, then can fold to regular torus. If klein bottle made of rubber, show how to fold.

Slice Klein bottle in half

Get 2 mobius bands glued at their edges

1. 3-d picture – horizontal slice
2. 2-d picture – slice diagonally from flip side to flip side

3-dimensional Analog of Klein Bottle

Glue 2 pairs of walls normally, and glue last with a side-to-side flip. Describe what happens when you walk through that wall.

Classwork

Handout

Homework

1. Read Chapter 4 and Chapter 5 in our text
2. Consider Euclidean geometry $(\mathbb{C}, G)$. Find 2 transformations T_1 and T_2 such that $(\mathbb{C}, G)/ < T_1, T_2 >$ gives the Klein bottle.
3. Does the projective plane contain a Mobius strip? If so, where is it? [Draw it]. If not, can you explain why not?

~~~~~~~~~~ Day 14 ~~~~~~~~~~

Projective Plane

1. made by glueing together diametrically opposite points on a sphere
2. can be realized by glueing together opposite points on rim of hemisphere
3. Topologically::: can also think of it as a disk with opposite points on edge identified (disk easier to draw than hemisphere).
4. Illustrate that projective plane is not orientable

Projective 3-Space
As above, can get projective 3-space by glueing opposite points of boundary of a 3-d ball.

Classwork
Handout

Chapter 5 – Connected Sums

Definition
The operation that takes 2 surfaces, removes a disk from each, and glues the surfaces together along the boundaries is called a "connected sum", and is indicated with a # symbol.

Examples

- Torus # genus 2 surface
- genus 4 surface # sphere

It's easy figure out what you get when you connect sum two orientable surfaces together – the genuses of the surfaces add.
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Connect Sum with Non-Orientable Summand

Removing a disk from a projective plane

1. Start with disk with antipodal points identified.
2. Remove disk from middle
3. Slice horizontally across, and mark edges that get glued
4. Get mobius band

Example

What is projective plane # projective plane? (Klein bottle)

Homework

No new homework.

~~~~~~~~~ Day 15 ~~~~~~~~~

Connect Sum with Non-Orientable Summand

Removing a disk from a projective plane

1. Start with disk with antipodal points identified.
2. Remove disk from middle
3. Slice horizontally across, and mark edges that get glued
4. Get mobius band

Example

What is projective plane # projective plane? (Klein bottle)

Notation

$\mathbb{E}^2$ or $\mathbb{R}^2$	Euclidean plane
$\mathbb{T}^2$	Torus
$\mathbb{K}^2$	Klein Bottle
$\mathbb{P}^2$	Projective Plane
D^2	Disk
$\mathbb{E}^3$ or $\mathbb{R}^3$	Euclidean 3-space
$\mathbb{T}^3$	3-Torus
$\mathbb{P}^3$	Projective 3-Space
D^3	Solid Ball
$\mathbb{E}^1$ or $\mathbb{R}^1$	Line
$\mathbb{S}^1$	Circle
I	Interval

Theorem

Every surface is a connected sum of tori and projective planes.
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Table of all possible surfaces

Create table with number of projective planes across top, and number of tori down side, starting with 0.

Exercise

Find the surface in the table which is topologically equivalent to

1. $K^2 \# P^2$
2. $K^2 \# T^2$
3. $K^2 \# K^2$

Theorem $T^2 \# P^2 = P^2 \# P^2 \# P^2$

Proof

1. Draw torus with disk removed, and $T^2\#$ mobius strip
2. Draw Klein bottle with disk removed, and $K^2\#$ mobius strip
3. Show one can be slid to the other
4. Glue on a mobius band to both.

Corollary

Therefore, every surface can be written as a connect sum of only tori, or a connect sum of only projective planes.

Exercise : Matching

Column A	Column B
$T^2 \# S^2$	$P^2 \# P^2$
K^2	$K^2 \# P^2$
$S^2 \# S^2 \# S^2$	$S^2 \# S^2$
$P^2 \# T^2$	$P^2 \# P^2 \# P^2 \# K^2$
$K^2 \# T^2 \# P^2$	T^2

Homework

Read Chapter 6

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Chapter 6 – Products

Motivation

- We know how to "add" two 2-D manifolds to get another 2-D manifold.
- Today we learn how to multiply manifolds!

Examples

- cylinder is product of circle and interval, because it is
    1. bunch of intervals arranged in a circle
    2. bunch of circles arranged (stacked) in an interval

  Write cylinder $\S^1 \times I$ "s-one cross I"
- a torus is the product of Circle with circle: $\S^1 \times \S^1$ (it's a circle of circles in 2 different ways)

Exercise
What is ...

1. $I \times I$
2. $\mathbb{R}^1 \times \mathbb{R}^1$
3. $\mathbb{R}^1 \times I$
4. $\mathbb{S}^1 \times \mathbb{R}^2$

Topology vs Geometry
We've only been discussing topological products. For a geometrical product, need

1. All circles are same size
2. All intervals are same size
3. The two items in the product are perpendicular to one another

Example

1. Draw pictures of cylinder that are, and are not, geometrical products
2. Geometrical $\mathbb{S}^1 \times \mathbb{S}^1$ is a flat torus.

3-manifolds

3-torus is circle of toruses (stack of papers), or a torus of circles (spaghetti)

Question:

Is it a geometrical product?

Exercise

Make 3-manifold by glueing front wall to back wall with side-to-side flip. It is a product. What is it a product of? (Klein bottle cross circle)

$\mathbb{S}^2 \times \mathbb{S}^1$

1. How would a flatlander deal with $\mathbb{S}^1 \times \mathbb{S}^1$? First imagine $\mathbb{S}^1 \times I$, then imagine inner edge glued to outer edge.
2. In geometric product, must imagine all circles same size
3. In geometric product, thread pulled tight will follow one of circles in annulus
4. Now describe $\mathbb{S}^2 \times \mathbb{S}^1$
5. Horizontal cross section is flat torus
6. Homogeneous: local geometry is everywhere the same
7. Isotropic: geometry is same in all directions.
8. $\mathbb{S}^2 \times \mathbb{S}^1$ homogeneous but not isotropic

Not a product.... $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$

Homework Read Chapter 7

~~~~~~~~~~ Day 17 ~~~~~~~~~~

Day off
~~~~~~~~~~

~~~~~~~~~~ Day 18 ~~~~~~~~~~

Chapter 7 – Flat Manifolds

1. View inside 3-torus
2. View inside $K^2 \times \mathbb{S}^1$
3. Glue with quarter turn
4. Show inside picture and guess glueing
~~~~~~~~~~

*** new office hours Th 12-1:30 (not W 2:30-4)

Möbius Geometry

(Recall) Definition

Let C^+ be the complex plane including ∞, and let M be the set of transformations of the form

$$Tz = \frac{az+b}{cz+d}$$

with a, b, c, and d complex constants such that $ad - bc \neq 0$. Such a transformation is called a *Möbius transformation* and the pair (C^+, M) models *Möbius geometry.*

(Recall) Properties

- Möbius geometry includes rotations, translations, rigid motions, homotheties, and inversion.
- Can write Möbius transformation as a composition of the above:

$$Tz = \frac{a}{c} - \frac{ad-bc}{c^2}\left(\frac{1}{z+\frac{d}{c}}\right)$$

  if $c \neq 0$.

(Recall) Theorem

Möbius geometry is actually a geometry (in other words, M is a group of transformations.)

Fixed point analysis

Get equation $cz^2 + (d-a)z - b = 0$ when $c \neq 0$, when $c = 0$ then analyze whether $a = d$.

Lemma

If T is not the identity transformation, then T has only one or two fixed points. A Möbius transformation with three or more fixed points must be the identity.

The Fundamental Theorem of Möbius Geometry

There is a unique Möbius transformation taking any three distinct complex numbers z_1, z_2, z_3 to any other three distinct complex numbers w_1, w_2, w_3.

Proof

Suffices to exhibit Möbius transformation taking z_1, z_2, z_3 to $1, 0, \infty$. Then if T sends z_1, z_2, z_3 to $1, 0, \infty$ and S sends w_1, w_2, w_3 to $1, 0, \infty$, the composition $U = S^{-1}T$ sends z_1, z_2, z_3 to w_1, w_2, w_3. Define

$$Tz = \frac{z - z_2}{z - z_3}\frac{z_1 - z_3}{z_1 - z_2}$$

and verify that T takes z_1, z_2, z_3 to $1, 0, \infty$.

For uniqueness, if V is another Möbius transformation taking z_1, z_2, z_3 to w_1, w_2, w_3, then the composition $V^{-1}U$ has three fixed points and is thus the identity.

Corollary

All figures consisting of three distinct points are congruent in Möbius geometry.

Definition

The *cross ratio* is the following function of 4 complex variables:

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$

If z_1, z_2, z_3 are held constant, then as a function of z_0 the cross ratio is the unique Möbius transformation sending z_1 to 1, z_2 to 0, and z_3 to ∞.

Example

Möbius geometry has enough transformation to take any 3 points to any other 3 points. What analogous result is true in translational geometry? Euclidean geometry?

Example

Find a Mobius transformation sending 1 to i, 0 to -3, and ∞ to 2

Homework

Chapter 5 Handout #1-8, 10, 11, (look at 14, 15)

~~~~~~~~~~ Day 20 ~~~~~~~~~~

Invariants of Möbius geometry

Angle measure since MT's are conformal

Cross ratio

- Proof of invariance of cross ratio: Let $Sz = \dfrac{z - z_2}{z - z_3}\dfrac{z_1 - z_3}{z_1 - z_2}$. Then ST^{-1} takes $Tz_1 \to 1$, $Tz_2 \to 0$, $Tz_3 \to \infty$. But the unique transformation taking $Tz_1 \to 1$, $Tz_2 \to 0$, $Tz_3 \to \infty$ is given by the cross ratio (z, Tz_1, Tz_2, Tz_3). Thus these transformations must be equal:

$$ST^{-1}z = (z, Tz_1, Tz_2, Tz_3).$$

Therefore

$$(z_0, z_1, z_2, z_3) = Sz_0 = ST^{-1}(Tz_0) = (Tz_0, Tz_1, Tz_2, Tz_3).$$

- Theorem: The cross ratio (z_0, z_1, z_2, z_3) is real if and only if the four points z_0, z_1, z_2, z_3 lie on a Euclidean circle or straight line.
- Proof: We will show that z is on Euclidean circle or straight line determined by z_1, z_2, z_3 if and only if cross ratio (z, z_1, z_2, z_3) is real. see book ****

Clines

- Definition: A *cline* is a Euclidean circle or straight line. (circles and lines are manifestations of a single geometric figure)
- Theorem: If C is a cline, and T is a Möbius transformation, then $T(C)$ is a cline.
- Proof: Choose 3 points on C. Then a point z is on C iff cross ratio is real. Cross ratio invariant, so cross ratio real iff cross ratio (Tz, Tz_1, Tz_2, Tz_3) is real. Cross ratio (Tz, Tz_1, Tz_2, Tz_3) real iff Tz lies on circle or line through Tz_1, Tz_2, Tz_3. Thus $T(C)$ is a cline.

Symmetry

- Definition: Let C be a cline passing through three distinct points z_1, z_2, z_3. Two points z and z^* are *symmetric* with respect to C if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$
~~~~~~~~~~

- Draw picture of two points symmetric with respect to line, circle.
- Symmetry is invariant of Möbius geometry. Describe what that means.

Homework

- Chapter 5 #1-8, 10, 11, (look at 14, 15) -18, 22, #25 extra credit
- Read Chapter 6
- Meet in afternoon for TeX tutorial

~~~~~~~~~~ Day 21 ~~~~~~~~~~

Steiner Circles

Motivation

- Steiner circles form a sort of coordinate system for Möbius transformations.
- Möbius transformations have either 1 or 2 fixed points. First, we'll study the simple cases, where the fixed points are at 0 and ∞. Then we'll use the action there to study case with arbitrary fixed points.

Possible cases – Canonical case

Discuss sensible coordinate system with lines thru origin, circles centered at origin.

- Two fixed point case:
  - Case 1: Elliptic transformation; simplest case = rotation about origin.
  - Case 2: Hyperbolic transformation; simplest case = homothety
  - Case 3: Loxodromic transformation; simplest case = homothety + rotation about origin.
- One fixed point case: Parabolic transformation; simplest case = translation (fixed point at infinity)

Two Fixed Points – General Case

Now think of two points p and q in the complex plane (these will be fixed points of the Möbius transformation), and consider the Möbius transformation taking p, q to $0, \infty$:

$$Sz = \frac{z-p}{z-q}.$$

The preimage of the coordinate system gives the Steiner circles.

Definition

The *Steiner circles of the first kind* with respect to p and q consist of all clines passing through both p and q. These are the preimages of lines through the origin.

Definition

The *Steiner circles of the second kind* with respect to p and q are all of the circles perpendicular to the Steiner circles of the first kind. These are the preimages of the circles centered at the origin.
~~~~~~~~~~

3 Types of 2-fixed-point MT's

For each type of Möbius transformation above, draw corresponding Steiner circles / action when p and q are fixed points, rather than 0 and ∞.

One Fixed Point – General Case

Move fixed point off to ∞ via $Sz = \frac{1}{z-p}$. Pre-image of grid gives *degenerate* Steiner Circles.

Normal form of a Möbius transformation

Important information about MT

Fixed points p and q, and "eigenvalue" λ. Normal form of a MT is a way to express the MT in terms of these important constants.

Getting to Normal Form – 2 Fixed Point Case

- Lift MT T to a transformation R in the "w" plane, where the fixed points are 0 and ∞ rather than p,q. The lifted transformation is given by

$$R = STS^{-1}.$$

- What does R look like? Well R is a MT, so it has the form

$$Rw = \frac{aw + b}{cw + d}.$$

  Since 0 and ∞ are fixed points, both b and c must be 0. So

$$Rw = \lambda w,$$

  where $\lambda = \frac{a}{d}$.

Definition

The *normal form* of T comes from the statement $ST = RS$: in 2 fixed point case it's

$$\frac{Tz - p}{Tz - q} = \lambda \frac{z - p}{z - q}.$$

Note

We still need exactly 3 constants to specify T.

Analysis of 2FP Möbius transformations depending on the constant λ

- Case $|\lambda| = 1$. Then $\lambda = e^{i\theta}$, so R is a rotation about the origin $\Rightarrow T$ is elliptic transformation. Draw action.
- Case λ real and positive. Then R is a homothety $\Rightarrow T$ is hyperbolic. Draw action.
- Case λ is general complex number: $\lambda = ke^{i\theta}$, $k \neq 1$ and $\theta \neq 0 \Rightarrow T$ is loxodromic. Try to draw action.

Getting to Normal Form – One fixed point case

- Move fixed point off to ∞ via $Sz = \frac{1}{z-p}$. Then $R = STS^{-1}$ has fixed point at ∞. Thus R is translation: $Rw = w + \beta$.
- Using $RT = TS$, we get
$$\frac{1}{Tz - p} = \frac{1}{z - p} + \beta.$$
This is the *normal form* of T in the one fixed point case.
- Draw *degenerate Steiner circles* (it's just a grid in the w plane), and describe action.

Homework

Chapter 6 #1-9

~~~~~~~~~ Day 22 ~~~~~~~~~

More on Mobius Transformations

Motivating Question

1. Recall $M = \{T : Tz = \frac{az+b}{cz+d},$ with $a, b, c, d \in \mathbb{C}, ad - bc \neq 0\}$
2. Recall that for every MT $Tz = \frac{az+b}{cz+d}$ there is a corresponding 2-dim matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
3. Question: Is it true that there is a 1-1 correspondence between MT's and matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$?

Definition of $SL_2\mathbb{C}$

$SL_2\mathbb{C} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbb{C}\}$

Called the "special linear group in dimension 2 over $\mathbb{C}$". Now we have a map $SL_2\mathbb{C} \to \{MT's\}$ which is 2:1.

Definition of $PSL_2\mathbb{C}$

$PSL_2\mathbb{C} = SL_2\,\mathbb{C} \,/ < \pm I >$ "Projective special linear group"

Hyperbolic Geometry

Disk Model

- Hyperbolic geometry consists of disk + group of all Möbius transformations preserving the disk.
- Definition: Let D be the unit disk in the complex plane. Let H be the set of transformations of D of the form
$$Tz = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}$$
where $|z_0| < 1$. The pair (D, H) models *hyperbolic geometry.*
~~~~~~~~~

Notes

- Hyperbolic geometry is a subgeometry of Möbius geometry
- Every figure in hyperbolic geometry has fewer congruent figures than in Möbius geometry, because hyperbolic geometry has fewer transformations.

Straight lines

- Definition of geodesic (need notion of distance)
- Description of hyperbolic straight line – chocolate pudding
- Pictures of hyperbolic straight lines

Hyperbolic Length in Disk model

Parameterized curves

- Ex: parametrize the line between two points.
- Ex: parametrize circle of radius r centered at a point a in the complex plane. First polar, then in form $x(t) + iy(t)$.
- In general, there is a function $x(t)$ and a function $y(t)$, and the curve γ is described parametrically as $\gamma(t) = (x(t), y(t))$.

Euclidean length of a (parametrized) curve

- Draw curve
- Show length as approximately sum of lengths of finite number of line segments.
- length(γ) between $t = a$ and $t = b$ given by

$$\text{length}(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b |z'(t)|\, dt$$

- Examples: calculating circumference of circle, length of given straight line

Definition Hyperbolic length (in disk model)
In the disk model of the hyperbolic plane, the *length* of a smooth curve parametrized by $|z(t)| = x(t) + iy(t)$ between points where $t = a$ and $t = b$ is

$$\text{length}(\gamma) = 2\int_a^b \frac{|z'(t)|\, dt}{1 - |z(t)|^2}.$$

~~~~~~~~~ Day 23 ~~~~~~~~~

Recall Definition Hyperbolic length (in disk model)

In the disk model of the hyperbolic plane, the *length* of a smooth curve parametrized by $|z(t)| = x(t) + iy(t)$ between points where $t = a$ and $t = b$ is

$$\text{length}(\gamma) = 2\int_a^b \frac{|z'(t)|\,dt}{1 - |z(t)|^2}.$$

Intuitive picture of hyperbolic distance

- Discuss effect of denominator on length
- Draw line segments and stick figures of same size at different places in disk.

Example of Calculating Length

Coming right up

Theorem

Length of a curve is an invariant of hyperbolic geometry; $l(T(\gamma)) = l(\gamma)$.

Proof

- Recall: A transformation of hyperbolic geometry has the form
$$Tz = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0}z}.$$
- Recall formula for length of γ:
$$\text{length}(\gamma) = 2\int_a^b \frac{|z'(t)|\,dt}{1 - |z(t)|^2}.$$
- Choose $w = T(z) = e^{i\theta}\frac{z-z_0}{1-\overline{z_0}z}$, and let γ be parametrized by $z(t)$.
- Then $T(\gamma)$ has parametrization $w(t) = T(z(t)) = ...$
- Start calculating length of $w(t)$:
- $w'(t) = ... = e^{i\theta}\,\dfrac{1 - |z_0|^2}{(1 - \overline{z_0}z(t))^2}\,z'(t)$
- So length of $T(\gamma) = ...$
~~~~~~~~~

1. make simple fraction
2. use $|a|^2 = a\overline{a}$ on denom and multiply out
3. cancel common terms and factor.
4. then part of numerator will cancel with denom

Definition Hyperbolic distance
If z_1 and z_2 are two points in the hyperbolic plane, then the *distance* between them, written $d(z_1, z_2)$, is the length of the hyperbolic geodesic segment between the two points.

Derive Distance Formula (disk model)

- Let z_1 and z_2 be in D.
- Apply a transformation from H which will send z_1 to 0, and rotate the image of z_2 to lie on the positive real axis, at r, say. Such a transformation is

$$Tz = e^{i\theta} \frac{z - z_1}{1 - \overline{z_1} z}.$$

- We'll find distance between images (OK since just proved length was invariant).
- Parametrize line and calculate distance $= \ln \left(\frac{1+r}{1-r} \right)$ (will need to use partial fractions)
- Substitute $r = \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right|$ to get formula

$$d(z_1, z_2) = \ln \left(\frac{1 + \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right|}{1 - \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right|} \right)$$

~~~~~~~~~ Day 24 ~~~~~~~~~

Recall Distance Formula

$$d(z_1, z_2) = \ln\left(\frac{1 + \left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right|}{1 - \left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right|}\right)$$

Theorem

These are the Fundamental properties of distance – necessary for distance to be useful.

If z_1, z_2, z_3 are points in the hyperbolic plane, then

1. $d(z_1, z_2) \geq 0$
2. $d(z_1, z_2) = d(z_2, z_1)$
3. If z_1, z_2, z_3 are collinear (in that order), then $d(z_1, z_2) + d(z_2, z_3) = d(z_1, z_3)$.

Theorem

If z_1 and z_2 are points in the hyperbolic plane, then the shortest curve connecting them is a hyperbolic straight line.

Corollary

Triangle inequality: If z_1, z_2, z_3 are any 3 points, then $d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$.

Parallelism

Parallel Postulate for Euclidean Geometry

There is exactly one parallel line through a point not on a given line.

Question

Is the same true for hyperbolic geometry?

Definition

The points on the unit circle are called *ideal points*

Definition
~~~~~~~~~

Two hyperbolic lines are *parallel* if they do not intersect inside D but do share one ideal point.

Definition

Two hyperbolic lines are *hyperparallel* if they do not intersect inside D and do not have an ideal point in common.

Examples

- Draw examples, including two hyperparallel lines which intersect (does not happen in Euclidean geometry).
- Draw pictures showing that parallel postulate does not hold.

Triangles

Show different triangles

Elliptic, Parabolic, Hyperbolic Transformations

Special Clines

- unit circle (acting as "line at infinity")
- clines perpendicular to unit circle (acting as straight lines)

Role of remaining clines Each is connected with either elliptic, parabolic, hyperbolic MT

- Definition: Let C be a portion of a Euclidean circle or straight line inside the unit disk, which is not perpendicular to the unit circle. Then C is a *cycle*.
- Definition: If C is entirely contained in D, then C is a *hyperbolic circle*.
- Definition: If C is tangent to the unit circle, then C is called a *horocycle*.
- Definition: If C intersects the unit circle, then C is a *hypercycle*.

~~~~~~~~~~ Day 25 ~~~~~~~~~~

Lemma

Let T be a Möbius transformation, C a circle, and z a point. If $T(C) = C$ and z is a fixed point of T, then the symmetric point z^* is also fixed by T.

Proof

Homework.

Classification of Transformations

Draw picture of each of 3 types of transformations, with their Steiner circles.

Apply lemma to unit circle, and transformations from H. Either

1. one fixed point inside and one outside unit circle
2. both fixed points on unit circle
3. only one fixed point (must be on unit circle)

Case 1: One point inside p, one point outside p^*

- Must fix unit circle $\Rightarrow$ elliptic transformation.
- unit circle is Steiner circle of second kind.
- Rotation of hyperbolic space, moving points around *hyperbolic circles* centered at p.
- Draw picture.
- Hyperbolic circle is curve traced out under elliptic transformation.

Case 2: Two fixed points on the unit circle

- must fix unit circle $\Rightarrow$ hyperbolic transformation.
- unit circle is Steiner circle of first kind (passes through fixed points)
- Translation of hyperbolic space
- mention axis of translation
- draw picture, hypercycles in picture
- Hypercycle is curve traced out by a point under hyperbolic transformation
~~~~~~~~~~

Case 3: One fixed point on the unit circle

- $\Rightarrow$ parabolic
- unit circle is degenerate Steiner circle
- degenerate Steiner circles inside unit circle are horocycles
- "parallel displacement" in hyperbolic space, where points move away and towards unique fixed point.
- draw picture with horocycles
- Horocycle is curve traced out by point under parabolic transformation.
- describe as degeneration of hyperbolic translation as two fixed points merge to one.

Note
Can think of a *cycle* as a curve (in any geometry) traced out when a point is repeatedly transformed by a single transformation. Thus hyperbolic geometry has 3 types of cycles.

Lobatchevsky's Formula

There is a fundamental relationship between angle measure and distance:

The Formula
Let the point p be a hyperbolic distance d from a hyperbolic straight line. If θ is the angle of parallelism of p with respect to this line, then

$$e^d = \tan\left(\frac{\theta}{2}\right).$$

Draw relevant pictures in both Euclidean and Hyperbolic space.

~~~~~~~~~~ Day 26 ~~~~~~~~~~

Upper Half Plane Model of Hyperbolic Space

We can use the inside of any cline to model hyperbolic geometry – how about the inside of the x-axis?

Definition
The *upper half plane* is the subset $U = \{z|\text{Im}(z) > 0\}$ of the complex plane.

Definition
Let $\overline{H}$ be the group of transformations of U of the form

$$Tz = \frac{az + b}{cz + d}$$

where a, b, c, d are *real* and $ad - bc > 0$.

Upper Half Plane
The pair $(U, \overline{H})$ models hyperbolic geometry.

Discuss and draw
geodesics, hyperbolic circles, horocircles, hypercircles

Discuss 3 types of transformations
elliptic, hyperbolic, parabolic

Theorem
In the upper half space model of hyperbolic space, length is given by

$$\text{length}(\gamma) = \int_a^b \frac{|z'(t)|}{y(t)}\, dt.$$

Proof
Show this gives same straight line distance between 2 points as in the disk model

- Start with p_1, p_2, draw line connecting them
- put in standard position: move p_1 to y axis, perform hyperbolic rotation so that connecting line goes to y axis.
~~~~~~~~~~

- Parametrize segment of y axis connecting points in y axis, and calculate length
- Show length $= \ln(y_1, y_2, \infty, 0) = \ln(p_1, p_2, r, s)$, same as in disk model.

Examples

- Draw line segment, calculate length.
- Calculate length of segment of unit circle

Hyperbolic Area

Area in Euclidean geometry

Recall area as a double integral:

- $A = \int\int_R dx\, dy$
- $A = \int\int_R r\, dr\, d\theta$

In Hyperbolic geometry...

Draw picture of small square to motivate definition of area in UHP (dx/y, dy/y)

Definition

The *area* of a region R in the upper half plane model of hyperbolic space is given by

$$A = \int\int_R \frac{dx\, dy}{y^2}$$

Examples

1. Find the hyperbolic area of the "unit" box with vertices (0,1), (0,2), (1,2), (1,1)
2. Find the hyperbolic area of the "unit" box with vertices (0,2), (0,3), (1,3), (1,2)

Classwork / Homework

Calculate the hyperbolic area of the

Areas of triangles

Doubly asymptotic triangle, with angle α

- Apply a MT......

  With angle α: put one ideal point at ∞, other at -1. Then area is

$$A = \int_{-1}^{\cos\alpha} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy\,dx}{y^2} = \int_{-1}^{\cos\alpha} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} - \arccos(x)\Big|_{-1}^{\cos\alpha} = \pi - \alpha$$

Trebly asymptotic triangle

has area π

General triangle

Use this to determine area of general triangle with angles α, β, γ: extend sides of triangle to boundary at infinity. Makes 3 extra triangles, with angles 0,0,$\pi = \alpha$, etc. Subtract areas to find area of inside triangle.

Definition

The difference $\pi = (\alpha + \beta + \gamma)$ is called the *angle defect* of a triangle. (explain terminology)

Theorem

The area of a triangle equals its angle defect.

Theorem

The sum of the angles of a triangle in hyperbolic geometry is less than π radians.

No Units

Explain why no units in hyperbolic space: because scaling makes a completely different object.

Definition

The *area* of a figure R in the disk model of the hyperbolic plane is defined by

$$A = \int\int_R \frac{4r\,dr\,d\theta}{(1-r^2)^2}$$

- general differential of length is
$$\frac{2|z'(t)|\,dt}{1-|z(t)|^2}$$
- polar coordinates; let $z = re^{i\theta}$
- $dA =$ (differential in r direction)(differential in θ direction)
- let parameter $t = r$, hold θ constant to get differential in r direction, etc

Theorem

If corresponding angles are equal in two triangles, then the triangles are congruent (in other words, there is no theory of similarity in hyperbolic geometry).

Proof

Sketch proof, putting both triangles together, with the coinciding angle at the origin.

Life in Hyperbolic plane

- unlimited parallel lines
- horocycles: all horocycles are congruent, but they are not straight!
- no squares or rectangles
- no similarity, so no maps or scale models

~~~~~~~~~ Day 27 ~~~~~~~~~

Announcements

- Show off day Friday May 10, 36 high school kiddies + faculty. In class. Faculty will come too.
- Explain project to others day, Thursday May 9, 12:25-2:05. 9 groups, each group gets up to 7 minutes, with 3 minute breaks in between, to grade and switch groups.
- Presentation order – choose number
- Write down order.
- No more homework to turn in

Topic Choice Order

1. Universe
2. * Euler Characteristic
3. Circle Pairs
4. 3-D geometry
5. * Geometry on Surfaces
6. Elliptic geometry

Video
Real Estate Opportunities

Remarks

1. Area of circle grows like e^r (colin did this calculation)
2. Sum of angles of triangle less than $180°$
3. As triangle gets smaller and smaller, gets more Euclidean, angle sum approaches 180.
4. Can have all right angle polygon of any number of sides > 4.

Question:
What regular polygons can you tile the Euclidean plane with?
~~~~~~~~~

1. parallelograms
2. some triangles
3. hexagons
4. no higher # sides

Question:
What regular polygons can you tile the hyperbolic plane with?
Any regular polygon with more than 4 sides.

~~~~~~~~~ Day 28 ~~~~~~~~~

Chapter 11 – Geometries on Surfaces

Draw 4 hexagons, give them geometry

Go around corners, Expand or contract so that lose cone angle. Can put geometry on it.
(also, projective plane hexagon)

Pairs of Pants

All surfaces of genus at least 2 can be made in hyperbolic space

1. Make a pair of pants out of two right angled hexagons (so must be hyperbolic)
2. Glue pairs of pants together to get surface
3. Can have homogeneous geometry by just gluing in mind, like flat torus, rather than actually glueing.
~~~~~~~~~

Chapter 12 – Gauss-Bonnet formula and Euler Characteristic

why

Euler characteristic is an easy-to-compute integer which tells you what geometry a surface admits.

Classification

Surfaces with....

- positive Euler Characteristic have elliptic geometry
- 0 Euler Characteristic have Euclidean geometry
- negative Euler Characteristic have elliptic geometry

Also tells whether surface is orientable or not, and what global topology is.

Definitions

0-d cell, 1-d cell, 2-d cell, Cell division

Exercise

Do a couple of cell divisions of sphere, and a couple of the torus. count vertices, edges, faces. be careful not to count vertices and edges twice.

Definition

$\chi = v - e + f$ is the Euler Characteristic

Homework

Read Chapters 11 and 12 in book.

Info about class summary

More on Euler Characteristic

Exercise
Do Euler characteristic of nonorientable surface, K^2, p^2

Exercise
Come up with formula for $\chi(A\#B)$, A and B surfaces

~~~~~~~~~ Day 29 ~~~~~~~~~

SnapPea

- Draw Knots
- Complement
- Fundamental Domain
- Horoballs
- Length Spectra

Curved Spaces

- ProgramFiles/Sample Spaces/Spherical/Binary6DL
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~~~~~~~~~ Day 30 ~~~~~~~~~

Gauss-Bonnet Formula

Proves that Euler Characteristic will be the same no matter what cell division you choose.

Gauss-Bonnet — Elliptic Formula

Given a cell division a surface with elliptic geometry, if A is the area of this surface, then

$$A = 2\pi(v - e + f) = 2\pi \cdot \chi.$$

Question

Why does this mean Euler characteristic is independent of cell division? (Because total area will be the same no matter what cell division, therefore $\chi = v - e + f$ will be the same no matter what cell division.)

Proof of GB, Elliptic case

Recall

- In hyperbolic geometry, Area of a triangle = angle defect = $\pi -$ sum of angles
- Similarly: In elliptic geometry, Area of triangle = angle excess = sum of angles - π

$\Rightarrow$ In elliptic geometry: (by splitting k-gon up into $k - 2$ triangles)
Area of polygon = sum of all angles $-(k - 2)\pi$.

Draw sphere and make cell division (some with triangles, some with more sides). Then

$$\begin{aligned}
\text{total area} &= [\text{Area 1st face of cell division}] + [\text{Area 2nd face}] + \cdots + [\text{Area nth face}] \\
&= [(\text{sum angles 1st face}) - (n_1 - 2)\pi] + \cdots + [(\text{sum angles nth face}) - (n_k - 2)\pi] \\
&= [(\text{sum angles 1st face}) + \cdots + (\text{sum angles nth face})] - (n_1 + \cdots n_k)\pi + (2 + 2 + \cdots + 2)\pi \\
&= [\text{sum of all angles}] - (n_1 + \cdots n_k)\pi + (2 + 2 + \cdots + 2)\pi \\
&= (2\pi)v - (2e)\pi + (2f)\pi \qquad \text{(because each edge is counted in 2 different (adjacent) cells)} \\
&= 2\pi(v - e + f)
\end{aligned}$$
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Gauss-Bonnet — Euclidean Formula

Given a surface with Euclidean geometry,

$$2\pi(v-e+f)=0.$$

Proof

Draw a torus, with a cell division. In Euclidean geometry, the sum of angles in a triangle is exactly π. Since a k-gon is constructed of $n-2$ triangles, the sum of angles of a k-gon is $(n-2)\pi$

$$\begin{aligned}
[(\text{sum angles 1st face}) + \cdots + (\text{sum angles nth face})] &= (n-2)\pi + \cdots + (n_k-2)\pi \\
2\pi v &= (n_1 + \cdots n_k)\pi - (2+2+\cdots+2)\pi \\
2\pi v &= 2\pi e - 2\pi f \\
2\pi(v-e+f) &= 0
\end{aligned}$$

Gauss-Bonnet — Hyperbolic Case

(*** If time is running out, state the theorem, say that the proof is very similar to the Elliptic case, and go straight to the general GB theorem ***)

Given a cell division a surface with hyperbolic geometry, if A is the area of this surface, then

$$-A = 2\pi \cdot \chi.$$

Proof

Similar argument to Elliptic case: Since area = angle defect = $\pi-$ sum of angles, we know

$$-A = \text{ sum of angles } - \pi.$$

So

$$\begin{aligned}
\text{-(total area)} &= -[\text{Area 1st face of cell division}] - [\text{Area 2nd face}] - \cdots - [\text{Area nth face}] \\
&= [(\text{sum angles 1st face}) - (n_1-2)\pi] + \cdots + [(\text{sum angles nth face}) - (n_k-2)\pi] \\
&= [(\text{sum angles 1st face}) + \cdots + (\text{sum angles nth face})] - (n_1 + \cdots n_k)\pi + (2+2+\cdots+2)\pi \\
&= [\text{sum of all angles}] - (n_1 + \cdots n_k)\pi + (2+2+\cdots+2)\pi \\
&= (2\pi)v - (2e)\pi + (2f)\pi \qquad \text{(because each edge is counted in 2 different (adjacent) cells)} \\
&= 2\pi(v-e+f)
\end{aligned}$$

General Gauss-Bonnet Formula

Puts all these formulas together in a single equation:

$$KA = 2\pi\chi$$

where k = constant Gaussian curvature $= -1, 0, 1$ depending on whether the surface is hyperbolic, euclidean, or elliptic

~~~~~~~~~~ Day 31 ~~~~~~~~~~

Announcements

1. Presentations tomorrow at 12:25 – please be there 5 minutes early. Manchester 206AB.
2. Tomorrow we grade presentations
3. Geometry Fair Friday at 10, Manchester 206AB
4. Will give out current grade on Friday
5. Have Jeff Weeks lectures on video if you want to check them out.
6.

Last time

Gauss-Bonnet Formula – $KA = 2\pi\chi$. Why does this imply that χ is invariant of the cell division?

Our Universe

3-D geometries

$E^3, S^3, H^3, S^2 \times R, H^2 \times R, \widetilde{SL_2}R$, Nil, Solv

What this means, JSJ Decomp

If a 3-manifold has a geometry, it's one of the above. But not all 3-manifolds admit a geometry.

JSJ Decomp

Geometrization Conjecture

Our Universe

Space seems to be relatively constant curvature, so we don't have to worry about this.

How to Detect Geometry of Universe

1. Einstein theory of Relativity, field equations didn't exactly work for static universe.
2. So he introduced Λ a fudge factor "Cosmological Constant"
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3. Friedman applied equations to expanding universe, and they worked exactly, without constant.
4. Einstein reluctantly agreed, but said that this had no practical application he coule think of.
5. Lemaître and Hubble found galaxy's rate of recession proportional to distance between us and galaxy. Got Hubble constant H, approx 7% per billion years
6. Model of expanding universe, raisins in muffin
7. Mean Density, compared to $D = \frac{3}{8\pi G}H^2$
   - High density ($\rho > D$) $\Leftrightarrow$ Elliptic
   - Borderline density ($\rho = D$) $\Leftrightarrow$ Euclidean Geometry
   - Low density ($\rho < D$) $\Leftrightarrow$ Hyperbolic Geometry
   - Know: $.02D < \rho < 4D$
8. Still doesn't exactly work, introduce Cos constant again, this time for "vacuum energy"
9. Studies indicate $\rho = .3D$
10. BOOMERANG project makes other measurements indicating space might be flat

~~~~~~~~~~ Day 100 ~~~~~~~~~~

Chapter 11 – Elliptic Geometry

1. Modeled in complex plane, use stereographic projection to get from plane to sphere
2. discuss rigid motions of sphere
3. discuss hairy ball theorem
4. S = set of all MT preserving diametrically opposite points; $T \in S$, then T can be put in either of following forms:
   - $Tz = \dfrac{az+b}{-\bar{b}z+\bar{a}}$ where $|a|^2 + |b|^2 = 1$
   - $Tz = e^{i\theta}\left(\dfrac{z - z_0}{\overline{z_0}z + 1}\right)$
5. compare with formulas for transformations of hyperbolic space.
6. Definition: The pair (C^+, S) models *elliptic geometry.* The set S is the *elliptic group.*
7. On sphere, shortest distance between 2 points is realized along *great circles*, circles on shpere whose center is in center of sphere.
8. Definition: In (C^+, S), a *great circle* is a circle C in the complex plane so that whenever C contains a point then it also contains the diametrically opposite point.
9. Definition: An *elliptic straight line* is an arc of a great circle.
10. NOTE: Two points do not determine a *unique* straight line in elliptic geometry.
11. Fix this problem by *identifying* diametrically opposite points. Get "single" elliptic geometry.
12. In "single" elliptic geometry, the great circles passing through diametrically opposite points become a bunch of straight lines through one point.
13. Definition: Let C/d be the complex plane with diametrically opposite points identified. The geometry $(C/d, S)$ is called *single elliptic geometry.* The geometry (C^+, S) defined previously is called *double elliptic geometry.*
14. justify how S can act on C/d.
15. points of single elliptic geometry correspond to diameters of sphere.
16. Theorem: Two distinct points determine a unique line in single elliptic geometry.
17. Proof: Distinct points z and w, correspond to 2 diameters, determines unique plane, determines great circle.
18. Describe disk model for single elliptic geometry.
19. Draw example of curve connecting two points in disk through an identified point on the boundary.
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20. Distance formula: $\text{length}(\gamma) = \int_z^b \frac{2|z'(t)|}{1+|z(t)|^2}\, dt$

21. Area: $A = \int\int_R \frac{4r\, dr\, d\theta}{(1+r^2)^2}$

22. Example: calculate length of a straight line across disk (should get π)

23. Example: area of a 2-gon angle α, double integral $dr\, d\theta$, get 2α

24. Compute area of triangle, creating 3 2-gons. Get $A = \alpha + \beta + \gamma - \pi =$ *angle excess.*

25. Theorem: The area of a triangle in single elliptic geometry is equal to its angle excess

26. Discuss a finite line segment can be produced to any length.

27. Explain why there are no parallel lines.

Homework

- Chapter 11 #1-2,4,6,8,13,14,16-18