Major Ideas in Calc 3 / Exam Review Topics

Here are some highlights of the things you should know to succeed in this class. I can not guarantee that this list is exhaustive!!!! Please be sure you are able to understand everything in your notes, and that you can work every assigned problem on your own, *without looking at the book or your notes*, before the exam. That will be the best way to prepare.

In General – Critically Important !!!

At all times, you should be absolutely certain whether the object you are dealing with is a function, an equation, a vector, a line, a point, a scalar, a surface, a plane, etc. If you are ever even the slightest bit confused about this, you can bet I will be able to tell this, and I won't think you have learned very much in this class!

Section 9.1 – Parametric Curves

- Know what a curve is
- Know how to parametrize functions, lines, circles, and ellipses

Section 9.2 – Calculus with Parametric Curves

- Know how to find the equation of a tangent line to a parametrically described curge
- Understand the formula for finding arc length of a curve
- Know how to find the arc length of any curve (hint: you'll have to parametrize first)

Section 10.1 – Three Dimensional Coordinate Systems

- Be able to locate any coordinate point on a graph of 3-space
- Know the distance formula in 3 dimensions
- Know the equation of a sphere

Section 10.2 – Vectors

- Know what a vector is
- Does a vector have a given starting point?
- Know what the magnitude of a vector is
- Know how to add, subtract, and take scalar multiples of vectors
- Know how to find a direction vector between two points
- Know how to find a unit vector in the direction of any vector
- Know what the standard basis vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ are
- Be able to use sin and cos to get the components of a vector given its length and angle with some fixed vector

Section 10.3 – The Dot Product

- Know two ways to calculate the dot product of two vectors, and when it makes sense to use either one
- Is the dot product of two vectors a number, or a vector?

- Know how to find the angle between two vectors using the dot product
- Know what the dot product measures
- Know some results of the definition of dot product (for example, when are two vectors perpendicular? What does this have to do with the dot product? And how can you see this must be true from the definition?)
- Know what orthogonal scalar projection is, and how to calculate it
- Know what orthogonal vector projection is, and how to calculate it

Section 10.4 – The Cross Product

- Know what dimension(s) the cross product is valid for
- Is the cross product of two vectors a vector or a number?
- Know what the cross product "means", both its direction and its length
- Know the definition of the cross product, the alternate definition, and at least one way to calculate using the alternate definition
- Know some facts about the cross product (ex: is $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$?

Section 10.5 – Equations of Lines and Planes

- Be able to write the vector equation of a line, given a variety of information. For example, you might be given:
  - two points on the line
  - a point on the line, and a vector in the direction of the line
  - in 2-D: a point on the line, and a vector perpendicular to the line, etc
- Know what two critical pieces of info you need to write the equation of a plane: a point on the plane, and a vector normal to the plane
- Be able to write the equation of a plane given those two pieces of info, or given any other info from which you can deduce those two pieces of info.

Section 10.6 – Cylinders and Quadric Surfaces

- Know what a cylinder and a quadric surface is
- Know what a trace, or cross-section, is
- Know how to use traces to graph **any** quadric surface
- Know how to complete the square!!!! This may be critical in your being able to identify the quadric surface!

Section 10.7 – Vector Functions and Space Curves

- Know what a vector valued function, or vector function, is
- Know how to find the derivative and integral of a vector function

Section 10.8 – Arc Length and Curvature in 3D

- Be able to calculate the arc length of any curve (same as before, but understand new notation for it)

Section 10.9 – Motion in Space: Velocity and Acceleration

- If you are given the position vector of an object, know how to calculate its velocity, speed, and acceleration
- Be able to use differentiation and integration to solve applied problems about projectiles, etc

Section 11.1 – Functions of Several Variables

- Know what a function of several variables is, and how it differs from a vector function
- Know what dimension the graph of a function lies in
- Be able to graph functions of 2 variables
- Know how to graph level curves of a function of 2 variables, and how to graph level surfaces of a function of 3 variables
- Be able to connect the graph of a function of several variables with the graph of its level sets, and be able to think in terms of one or the other depending on what is appropriate for the problem
- Pay attention to "degenerates"

Section 11.3 – Partial Derivatives

- Understand what a partial derivative means
- Be able to tell if the sign of any partial derivative is positive or negative from a graph of the function
- Be able to tell if the sign of any partial derivative is positive or negative from a graph of the level sets of a function
- Be able to take the partial derivatives of a formula of a function quickly and correctly, with absolutely no mistakes.
- Understand the picture I've been drawing of the graph of a function of 2 variables, which involves traces on the graph, and the slopes of those traces at a given point. Know what this picture has to do with partial derivatives.
- Know how to take second partial derivatives
- Understand whether f_{xy} means to take the partial derivative with respect to x or y first
- Understand whether $\dfrac{\partial^2 f}{\partial x \partial y}$ means to take the partial derivative with respect to x or y first
- Know Clairaut's Theorem
- Know what a partial differential equation is
- Be able to show whether a given function satisfies a given partial differential equation.

Section 11.4 – Tangent Planes and Linear Approximations

- Given a *function* of 2 variables, be able to write the equation of the tangent plane to the graph of that function at a given point by simply using the partial derivatives of the function.
- Note that the technique above does not work if you are given the *equation* of a surface! These are different objects!
- Given a function of several (any number of) variables and a point, be able to write the formula for the linear approximation of that function at that point.
- Be able to use your linear approximation to approximate values of that function at nearby points.

Section 11.5 – The Chain Rule

- Understand that when you take the derivative of a composition of functions, then the rate of change of the composition is the product of the rates of change of the individual functions.
- Be able to produce the appropriate chain rule to find the derivative of any combination of compositions of multivariable functions and vector valued functions.

Section 11.6 – Directional Derivatives and the Gradient Vector

- Know what the gradient vector is, and how to calculate it
- Know what sorts of functions you can take the gradient of (hint: not all functions have a gradient vector!)
- Understand what a directional derivative is
- Understand the picture we've drawn several times in class illustrating what a directional derivative is, and how the formula comes about
- Know how to calculate the directional derivative, and know what the directional derivative has to do with the gradient vector.
- Know whether the directional derivative is a vector or a scalar!
- Know the dimension of the gradient vector of a function at a point, and know where it makes sense to locate the gradient vector on the graph of a function
- Know how to find which direction to head in from a point in order to increase (or decrease) the value of a function the most quickly. Understand what the formula for directional derivative has to do with this
- Know the connection between the gradient vectors and the level sets of a function.
- Know what the function is doing in a direction perpendicular to the gradient vector
- Be able to find the maximum rate of change of a function at a given point. And be sure you know why there is more than one rate of change of a multivariable function at a point!

Section 11.7 – Maximum and Minimum Values

- Know what a local max, local min, and saddle point are
- Know the definition of critical point of f: where $\nabla f(x)$ doesn't exist or $\nabla f(x) = 0$
- Know to find critical points of a function (need either *one* partial doesn't exist, or *all* partials =0
- Be able to classify critical points as local maxes, mins, saddle using second derivative test
- Know what level curves, gradient vectors look like near a max, min, and saddle point

Section 11.8 – LaGrange Multipliers

- Know what it means to find the max or min of a function f subject to a given constraint $g(x, y) = c$
- One technique: parametrize constraint curve, plug into function, then do normal 1-variable max-min problem
- Understand how ∇f compares to ∇g at a max or min of f subject to the constraint $g(x, y) = c$
- Given a graph of the level sets of f and the graph of the constraint $g(x, y) = c$, be able to locate maxes / mins of f subject to the constraint
- Know how to use Lagrange multipliers to maximize or minimize functions subject to some constraint. (crit points of f *subject to the constraint* $g(x, y) = c$ are the points where $\nabla f = \lambda \nabla g$
- Know how to find absolute maxes and mins of a function on closed regions with boundary:

1. Find the candidates
   - find crit points inside region (section 11.7)
   - find crit points on the boundary of region
     (a) endpoints
     (b) corners
     (c) points where constraint curve is tangent to level curve of function (Lagrange multipliers)
2. Test the candidates:
   - find values of f at all these points: biggest is max, smallest is min

Section 12.1 – Double Integrals Over Rectangles

- As a pre-requisite for the rest of the semester, know how to evaluate any integral in sight.
- Know how to estimate the double integral of a 2-variable function from a graph of the level curves of the function
- Know that the double integral of a function over a region in the plane can be interpreted as the volume under the graph of the function

Section 12.2 – Double Integrals Over General Regions

- Be able to set up a double integral of any function over any given region in the plane
- Know how to evaluate integrals using : parts, "u" substitution, trig integrals, trig substitution
- Know how to draw the region of integraion given a double integral
- Know how to reverse order of integration (you *must* draw the region first)
- Know that sometimes to evaluate an integral, you have to switch the order of integration
- Be able to do all problems on the integral handout
- Know that the integral of a density function over a region gives you the total mass of the region
- Remarks
  - limits on outer integral must be constants
  - can have outside variable in inside limits
  - endpoints of integration come from *region* you're integrating over
  - what you actually integrate is the *function* you're asked to integrate
  - for some *regions*, either integrating one way or the reverse is more logical.
  - for some *integrands*, can only evaluate one way

Section 12.3 – Double Integrals in Polar Coordinates

- Be able to convert an integral from rectangular coordinates to polar coordinates (must draw region!)
- Be able to set up double integral of polar coordinates of any function over any region in the plane (even over non-round regions)
- Know that sometimes if you can't evaluate an integral, you will be able to evaluate it if you switch to polar coordinates

Section 12.5 – Triple Integrals

- Be able to set up and evaluate triple integrals in rectangular coordinates over any given region in 3-dimensional space
- Remarks
  - limits for outer integral must be constants
  - limits for middle integral can involve only outer var
  - limits for inner integral can involve 2 outer vars

Sections 12.6 and 12.7 – Triple Integrals in Cylindrical and Spherical Coordinates

- Be able to convert any rectangular integral into cylindrical coordinates
- Remember $dx\ dy\ dz \mapsto r\ dr\ d\theta\ dz$
- Be able to convert any rectangular integral into spherical coordinates
- Remember $dx\ dy\ dz \mapsto \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$
- Be able to evaluate any integral you've set up in either cylindrical or spherical coordinates
- Given a region in 3D, be able to decide whether it makes most sense to set up an integral over that region in rectangular, cylindrical, or spherical coordinates.
- Given an integrand in rectangular coordinates, be able to decide whether that integrand would be easier to integrate if translated into spherical or cylindrical coordinates
- Given a 3D integral, be able to draw / identify the region in space that the integration is to be performed over!
- In general, know how to set up triple integrals on regions in space such as cylinders, spheres, intersections of these, solids bounded by graphs of functions and planes... using rectangular, spherical, or cylindrical coordinates where appropriate

Section 13.1 Vector Fields

- Know what a vector field is, in the plane and in 3-space
- Know how to sketch a vector field in the plane, given a formula for the vector field
- Be able to match equations of vector fields with their graphs
- Be able to sketch the gradient vector field for a function of several variables, and understand how to spot maxes and mins of the function from this vector field.
- know what the phrases "conservative vector field" and "potential function" mean

Section 13.2 –Line Integrals

- be able to parametrize straight lines, circles, graphs of 1-dimensional functions
- Determine whether your parametrization of the curve is orientation preserving or reversing.
- Know the effect of orientation of parametrizations, orientation of curves on path integrals and line integrals
- two types of line integrals: one for real valued functions, one for vector fields.
- In general, be able to set up and evaluate a line integral over any curve (integrating real valued function or vector field)
- line integral of real valued function: $\int_C f(x,y)ds = \int_a^b f(\overrightarrow{r}(t))\ \ ||\overrightarrow{r'}(t)||\ dt$

- line integral of vector field: $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$
- know how to calculate both types of line integrals (remember: must parametrize!!)
- know what both types of line integrals mean – what are they measuring or calculating?
- for line integral of vector field, know what kind of curve will give
  - highest line integral
  - 0 for line integral
  - lowest line integral
- alternate notation for line integral of vector field: $\int_C \vec{F} \cdot d\vec{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz$

Section 13.3 – Fundamental Theorem for Line Integrals

- know what is meant by the phrase "independence of path"
- Know what conservative vector field means
- Know how to show a vector field is or is not conservative
- know how to find a potential function for a conservative vector field by "partial antidifferentiation"
- Know all 4 properties of conservative vector fields
- Know shortcuts for evaluating integrals of conservative vector fields over curves
- Fundamental Theorem of Line Integrals
  - Know the statement of the theorem, and what dimensions it applies to
  - Know when, and how, to apply the theorem.
  - Realize that the theorem has *conditions*, and you can only apply it if your situation satisfies those conditions!
- The following are equivalent for a vector field F *where all partials exist*
  - F is a conservative vector field
  - For any oriented simple closed curve C, $\int_C \vec{F} \cdot d\vec{r} = 0$
  - For any two oriented curves C_1 and C_2 which have the same endpoints, $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$
  - F is the gradient of some function f
- Know Theorem 6, and realize that there are **3 things to show** before you can apply this theorem!

Section 13.5 – Curl and Divergence

- know what the symbol ∇ stands for in 2 and 3 dimensions: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
- divergence : div $F = \nabla \cdot F$
  - know how to calculate divergence of a vector field in 2 and 3 dimensions
  - know that divergence is rate of expansion of vector field
  - know how to recognize whether a vector field has positive, negative, or 0 divergence just by looking at its graph
  - note divergence is a scalar!!!
- curl : curl $F = \nabla \times F$ (3D) or curl $F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ (2D)

- curl F measures the amount of rotation that a small paddle wheel placed in the vector field would experience at a given point
- curl F is a vector in 3d : magnitude = strength or speed of rotation; direction points along axis of rotation, pointing in direction given by the right hand rule
- know how to recognize whether a vector field has positive, negative, or 0 curl just by looking at its graph

- curl of a gradient : $\nabla \times (\nabla f) = 0$
- divergence of a curl : $\nabla \cdot (\nabla \times F) = 0$
- Know the statement of the Curl Test, and **know that there are conditions to check** before you are allowed to apply this test!!!
- know how to apply the curl test to determine whether a vector field is conservative or not.
- Overview: the main theorems from Chapter 13 discusses variants of FTC for multivariable calculus, and look like

$$\int_{\text{region}} \text{derivative of } f = \int_{\text{boundary of region}} f$$

where derivative = gradient, divergence, or curl.

Section 13.4 – Green's Theorem

- Know what the positive orientation of a simple closed curve is
- Know what orientation has to do with Green's Theorem
- Know 2 statements of Green's Theorem:
  - Let F be a v.f. on a region D in the plane. Then $\int_{\partial D} \vec{F} \cdot d\,\vec{r} = \int\int_D (\text{Curl}F) \cdot \vec{k}\; dx\,dy$
  - If $F(x, y) = (P(x, y), Q(x, y))$, D a region in the plane, then $\int_{\partial D} P\,dx + Q\,dy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$
- Know what dimension Green's Theorem applies to
- Know what
- Know why Green's Theorem makes sense by interpreting the symbols
- Know how and when to apply Green's Theorem
- Realize that the theorem has *conditions*, and you can only apply it if your situation satisfies those conditions!

Section 13.6 – Parametric Surfaces and Their Areas

- be able to parametrize a wide variety of surfaces, including but not limited to: planes, spheres of all sizes, hemispheres, cylinders along any axis, graphs of 2-variable functions, cones, paraboloids, ellipsoids
- be able to use a parametrization of a surface to write the equation of the tangent plane to the surface
- Summary of Notation:
  - $r : \mathbb{R}^2 \to \mathbb{R}^3$ is parametrization of surface (2 input vars, 3 output vars)
  - $d\,\vec{S}$ = surface element = normal to bit of surface = $r_u \times r_v\; du\; dv$
  - $dS = |d\,\vec{S}\,|$ = surface *area* element = $\|r_u \times r_v\|\; du\; dv$
  - Surface Area = $\int\int_S dS = \int\int_D \|r_u \times r_v\|\; du\; dv$,

Section 13.7 – Surface Integrals

- Two types of surface integrals: one for real valued functions, one for vector fields.
- Integrating real valued function over surface: $\int\int_S f dS = \int\int_D f(\vec{r}\ (u,v)) \| \vec{r}_u \times \vec{r}_v \| \, du\, dv$
- Surface Integral of vector field: $\int\int_S \vec{F} \cdot d\vec{S} = \int\int_D \vec{F}\ (\vec{r}\ (u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ du\ dv$
- Surface integrals of vector fields measure how much vector field is flowing through a surface – rate of flow of F across S. Also called flux.
- Instead of dot product with tangent vector (as in line integrals of vector fiends), surface integrals of vector fields use dot product with normal vector to surface
- know what an orientation of a surface is
- be able to produce an orientation to a surface from your parametrization of the surface
- be able to tell whether your parametrization is orientation preserving or reversing, and know what to do to calculate flux in either case
- Know the effect of orientation of parametrizations, orientation of surfaces on integrals of both scalar functions and vector fields
- In general, be able to set up and evaluate a surface integral of either a scalar function or a vector valued function over any surface

Section 13.8 – Stokes' Theorem

- know what the boundary of a surface is
- Know what dimension Stokes' theorem applies to
- Stokes' theorem (3D): Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a vector field, and D a surface in $\mathbb{R}^3$. Then

$$\int_{\partial D} \vec{F} \cdot d\,\vec{s} = \int\int_D (\text{Curl } F) \cdot d\,\vec{S}$$

- Note: must orient D and ∂D coherently, using right hand rule !!!
- Remember, if you can't evaluate some surface integral of the curl of some function, try replacing it with another surface with the same boundary (using Stokes' theorem 2 times)
- Know why Stokes' Theorem makes sense by interpreting the symbols
- Know how and when to apply Stokes' Theorem
- Realize that the theorem has *conditions*, and you can only apply it if your situation satisfies those conditions!

Section 13.9 – Gauss' Divergence Theorem

- Know what the boundary of a solid is
- Divergence theorem in space: $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, S is closed surface (no boundary, no "holes"), W = 3D region it bounds. Then

$$\int\int_{\partial W} \vec{F} \cdot\ d\vec{S} \quad = \quad \int\int\int_W \text{div}\ \vec{F}\ dV$$

amount gas leaving W through ∂W (flux across W) = amount gas produced in W

- Know why Divergence Theorem makes sense by interpreting the symbols
- Know how and when to apply Divergence Theorem
- Realize that the theorem has *conditions*, and you can only apply it if your situation satisfies those conditions!