Practice Proofs I

The best way to learn to write proofs is lots of practice. I love doing proofs, and I would love look at every one of your attempts at these problems in my office hours. If you really want to feel comfortable writing proofs, then you should try to do every one of these. None of them are to be turned in, so it will be to your great disadvantage to just go look them up in a book. You learn by struggling with the problems.

If it's not specified in the problems below, V will be a vector space over F (this means that F is the set of scalars for V).

1. Prove that if A is a square matrix, then $A + A^T$ is symmetric.
2. Prove the cancellation law for addition: If x, y, and z are elements of V such that $x + z = y + z$, then $x = y$.
3. Prove that the zero vector in V is unique. (To do this, prove that if $\mathbf{0}$ and $\mathbf{Z}$ both satisfy property 3, then $\mathbf{0} = \mathbf{Z}$
4. Prove that additive inverses are unique (part of this problem is to figure out what it means to prove this. See previous problem for hints).
5. Prove that $0\mathbf{x} = \mathbf{0}$ for each $x \in V$. (note that the first zero is a scalar in F, and the second zero is the zero vector).
6. Prove that $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ for each scalar a and each vector $x \in V$.
7. Prove that $a\mathbf{0} = \mathbf{0}$ for each scalar $a \in F$.
8. Let $V = \{\text{differentiable functions } f : \mathbb{R} \to \mathbb{R}\}$, and $F = \mathbb{R}$ (the scalars). Prove that V is a vector space under the usual operations of addition and scalar multiplication that you learned in calculus.
9. Let $V = \mathbb{R}^2$. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define
$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \qquad \text{and} \qquad c(a_1, a_2) = (ca_1, a_2).$$
Is V a vector space under these conditions? Prove or disprove.
10. Let $V = \mathbb{Z}^4$ (4 component vectors with integer coordinates) and let $F = \mathbb{R}$, the real numbers. Is V a vector space over F, with the usual coordinatewise addition and scalar multiplication? Prove or disprove.
11. Let $V = \{2 \times 2 \text{ matrices with real number entries}\}$ and let $F = \{\text{ rational numbers}\}$. Is V a vector space over F under the usual definitions of matrix addition and scalar multiplication? Prove or disprove.
12. Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$ define
$$c(a_1, a_2) = (a_1, 0).$$
Is V a vector space under these operations? Justify your answer.
13. Are the following subsets of $\mathbb{R}^3$ also subspaces of $\mathbb{R}^3$? Prove or disprove.
    (a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

(b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$

14. Let c_0 be a fixed constant in $\mathbb{R}$, and let $\mathcal{F}$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that $W = \{f : \mathbb{R} \to \mathbb{R} | f(c_0) = 0\}$. is a subspace of $\mathcal{F}$.

15. Prove that a subset W of a vector space V is a subspace of V if and only if $\mathbf{0} \in W$ and $ax+y \in W$ whenever $a \in F$ and $x, y \in W$.

16. Let W_1 and W_2 be subspaces o a vector spave V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

17. Let W be a subspace of V. Prove that all linear combinations of vectors from W are vectors in W: if $w_1, \cdots, w_n \in W$ and $c_1, \cdots, c_n \in F$, then $c_1 w_1 + \cdots + c_n w_n \in W$.