LaTeX Class Review by *

While starting from the beginning may be helpful, I feel it will be more helpful to me, since I plan on using this as a study guide, to start a little further into the course (bypassing ideas that we have learned in other classes like 1-1 and onto, and bypassing some basic ideas such as what a topology is, the definition of an open set, continuity, etc.) focusing on what I feel are the more complicated ideas and providing examples. Also, while defining complicated ideas, I will treat them the way we did the definition quiz in class, defining all the key components that have yet to be defined in these notes.

Definition: Basis - A basis β for a topology T on a space X is a set so that every set in T can be obtained by taking arbitrary unions and finite intersections of elements of β.

Definition: Homeomorphic - Two spaces are homeomorphic if there exists a **homeomorphism** between them.

Note: A homeomorphism is a bijection between points in X and points in Y and between open sets in X and open sets in Y.

Definition: Homeomorphism - A homeomorphism between two spaces X and Y exists if there exists a **bijection** $f : X \to Y$ such that f and f^-1 are both continuous.

Definition: Bijection - $f : X \to Y$ is a bijection if it is one-to-one and onto.

Definition: Constant Map - $C : X \to Y$ is considered a constant map if there exists a $c \in Y$ such that for all $x \in X$, $C(x) = c$.

Note - A function $f : X \to Y$ is called constant if there is some $c \in Y$ so that $f(x) = c$ for every $x \in X$.

Definition: Homotopic - Two maps $f, g : X \to Y$ are homotopic if there exists a map $F : X \times I \to Y$ such that for each $x \in X$,

$$F(x, 0) = f(x)$$

$$F(x, 1) = g(x).$$

We write $f \simeq g$ or F is a homotopy between f and g.

Note: Homotopic paths must follow third and fourth properties, that

$$F(0, t) = x_0$$

$$F(1, t) = x_1.$$

If the paths are loops then $x_0 = x_1$. [Remember, the path/loop is the actual map, not the image.]

Nulhomotopic - The function $f : X \to Y$ is nulhomotopic if it is homotopic to a constant map.
Additional Thought: Two maps $f, g : (X, A) \to (Y, B)$ are **homotopic rel** A, written as $f \sim g$ rel A, if there exists a homotopy $F : X \times I, A \times I) \to (Y, B)$ that follows the first two criteria of a homotopy and if for all $x \in A$

$$F(x, t) = F(x, 0)$$

The idea is that during the homotopy the points in A do not move, they remain constant thought out the homotopy.

Definition: Contractible - A space is called contractible if it is homotopic equivalent to a point

Lemma : Regarding Contractible Spaces [Proved in Class] - Let X be any space and let p be any arbitrary point. If $X = p$ then $X \simeq p$.

Definition: Homotopy Equivalent - Two spaces X and Y are homotopy equivalent if there exists maps $f : X \to Y$, and $g : Y \to X$ such that

$$g \circ f \simeq I_X$$

$$f \circ g \simeq I_Y$$

[Where I_X is the identity map $I_X : X \to X$ by $I_X(x) = x$ and $I_Y : Y \to Y$ by $I_Y(y) = y$].
Example of Homotopy Equivalent: Prove the unit disk in $\mathbb{R}^2$ is contractible [Aka - homotopy equivalent to a point].

Proof: - Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and p be some arbitrary point in $\mathbb{R}^2$. Define:

$$f : D \to \{p\} \text{ by } f(x, y) = p \forall (x, y) \in D.$$

Further define:

$$g : \{p\} \to D \text{ by } g(p) = (0, 0).$$

Consider the function $f \circ g : \{p\} \to \{p\} = f \circ g(p) = p$.
Notice that this is the same as $I_{\{p\}} : \{p\} \to \{p\}$ by $I\{p\}(p) = p$.
From the lemma above, we know $f \circ g \simeq I_{\{p\}}$.
Now, consider $g \circ f : D \to D$, we know

$$g \circ f(x, y) = g(f(x, y)) = g(p) = (0, 0).$$

Also, consider that $I_D(x, y) = (x, y)$.
Define: $F : D \times I \to D$ by $F((x, y), t) = (x(1 - t), y(1 - t))$. Note, this straight line homotopy exists entirely in D due to our definition of D. Now,

$$\begin{aligned} F((x, y), 0) &= (x(1 - 0), y(1 - 0)) \\ &= (x, y) \\ &= I_D(x, y) \end{aligned}$$

$$\begin{aligned} F((x,y),1) &= (x(1-1), y(1-1)) \\ &= (0,0) \\ &= g \circ f(x,y) \end{aligned}$$

Also, F is built on continuous polynomials so it too is continuous.
Thus $I_D \simeq g \circ f$. Therefore D is contractible [homotopy equivalent to a point] □

Definition: Convex - Let C be an arbitrary space. C is convex if and only if for all $x, y \in C$, the straight line segment connecting them is contained entirely in C. [C is convex iff for all $x, y \in C$ the set $\{tx + (1-t)y : t \in [0,1]\}$ is a subset of C]

Definition: Homotopy classes of loops - Let α, β be loops with $\alpha(0) = \beta(0)$ then $\alpha \simeq \beta$ rel ∂I if and only if there exists a map $f : I \times I \to X$ such that

$$F(s,0) = \alpha(s)$$

$$F(s,1) = \beta(s)$$

$$F(0 \text{ or } , t) = \alpha(0)$$

.

Homotopy Class of Loop α based at $x_0 \in X$. -

$$[\alpha] = \{\beta : (I, \partial I) \to (X, x_0) | \alpha \simeq \beta \text{ rel } \partial I\}$$

Definition: Constant Loop - The constant loop C in space X based at $x_0 \in X$ is defined as

$$C : (I, \partial I) \to (X, x_0) \text{ by } C(s) = x_0.$$

Definition: Group - a set [G] with an operation [*] that acts on the set $* : G \times G \to G$ with the three properties: 1. **identity** 2. **inverse** 3. **associative**.

Definition: Identity - In set G, with operation $\cdot$, there exists $e \in G$ such that for all $g \in G$, $e \cdot g = g = g \cdot e$.

Definition: Inverse - In set G, with operation $\cdot$, for all $g \in G$, there exists $h \in G$, such that $g \cdot h = e = h \times g$

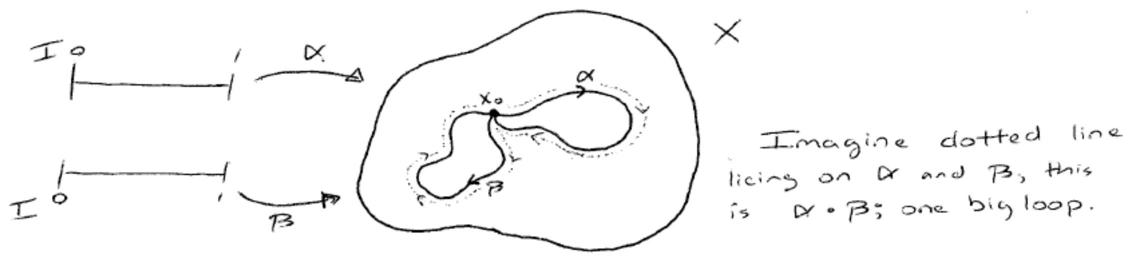
Definition: Associative - In set G, with operation $\cdot$, for all $g, j, h \in G$, $(g \cdot h) \cdot j = g \cdot (h \cdot j)$.

Fundamental Group of X based at $x_0 \in X$ **-** $\pi_1(X, x_0)$ is the set of all homotopy classes of all loops based at x_0, fully defined below. Let $\alpha, \beta : (I, \partial I) \to (X, x_0)$ where $\alpha \cdot \beta : (I, \partial I) \to (X, x_0)$ by $\alpha \cdot \beta(s)$ as defined below.

[Group] - $\pi_1(X, x_0) = [\alpha] : \alpha : (I, \partial I) \to (X, x_0)$

[Operation] - $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ and

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & \text{for } s \in [0, .5] \\ \beta(2s-1) & \text{for } s \in [.5, 1] \end{cases}$$



[Take note that we can only define multiplication of loops if they have the same base points. Also, the idea behind $\alpha \cdot \beta$ is that is spends the first 1/2 of the time mapping α and the second 1/2 of the time mapping β making one big continuous loop.]

Note: In class we proved that for all $x \in X$, $\pi_1(X, x_0)$ is a group [aka-the fundamental group is in fact a group]. Had to prove it satisfied the three properties to be a group, and the operation had to be **well-defined** and continuous.

Definition: Well-defined - If $[\alpha_1] = [\alpha_2]$ and $[\beta_1] = [\beta_2]$ then

$$[\alpha_1 \cdot \beta_1] = [\alpha_2 \cdot \beta_2].$$

Prop 17.1 - Used to prove the fundamental group operation is continuous.

Let $f : X \to Y$ and $X = A \cup B$. If f rel A and f rel B are continuous, the f is continuous.

Definition: Homomorphic - Two groups with operations $(G, *_g)$ and $(H, *_h)$ are homomorphic if there exists a function $f : G \to H$ such that for all $g_1, g_2 \in G$

$$f(g_1 *_g g_2) = f(g_1) *_h f(g_2).$$

Definition: Isomorphic - $[G \cong H]$ - Two groups with operations $(G, *g)$ and $(H, *h)$ are isomorphic if there exists a function $f : G \to H$ that is a bijective homomorphism.

Fact: Any space X is **simply connected** if it is path connected and the fundamental group of X is isomorphic to the trivial group. So, far all $x_0 \in X$,

$$\pi_1(X, x_0) \cong \mathbf{\{1\}}.$$

Trivial Group - {1} - The trivial fundamental group, commonly denoted as $\{1\}$, is equal to the constant loop.

$$\pi_1(X, x_0) = [\alpha] : C : (I, \partial I) \to (X, x_0)$$

.

Theorem - $\pi_1(S_1, (1, 0)) \cong \mathbb{Z}$

Lemma : Unique Path Lifting Lemma[Proved in Class] - Let $\alpha : (I, \partial I) \to (S_1, (1, 0))$ and $\pi\mathbb{R} \to S_1$ by $\pi(r) = (\cos 2\pi r, \sin 2\pi r)$. There exists a unique path $\widetilde{\alpha} : I \to \mathbb{R}$ with $\widetilde{\alpha}(0) = 0$ and for all $s \in I$, $\pi \circ \widetilde{\alpha}(s) = \alpha(s)$.

Here is the commutative diagram. [In the diagram $w = \pi(r)$.]

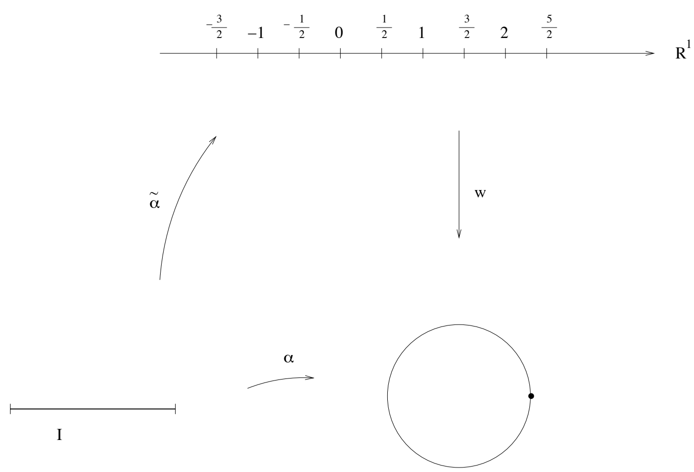


Figure 1: Lifting a loop from the circle to the real line.

Note: The Unique Path Lifting Lemma was used to prove $\pi_1(S_1, (1, 0)) \cong \mathbb{Z}$. To prove this lemma we used the idea of local homeomorphism. Ideally we would like $\widetilde{\alpha} = \pi^{-1} \circ \alpha$ but π in not invertible. However, it is invertible rel A, where A is an interval in $\mathbb{R}$ of length < 1. We defined a u and v so that $u = S^1 - (-1, 0)$ and $v = S^1 - (1, 0)$, thus u and v are open sets where $u \cup v = S^1$, and $\alpha^{-1}(u)$ and $\alpha^{-1}(v)$ are open sets and $\alpha^{-1}(u) \cup \alpha^{-1}(u) = I$. This breaks I into segments so that the interval $\alpha([t_i, t_{i+1}, ...]) \subseteq u$ or v. The function $f : \pi_1(S_1, (1, 0)) \to \mathbb{Z}$ maps loops to integers. It is a function which associates α with the number of times the image of α wraps around S^1.

Definition: Strong Defamation Retraction (sdr) - Let $A \subseteq X$ and the function $r : X \to A$ be a **retraction**. r is a sdr if $I_x \simeq r$ rel A.

There is a sdr if you can formulate(or picture) a function that smoothly moves all points in X onto A. [Think homework]

Retraction of X onto A - A continuous function $r : X \to A$ with $r/A = I_A$. Can be understood as, $r \circ i = I_A$. The idea is that elements outside of A get moved onto A and elements of A are fixed.

$$\begin{array}{rccc} & A & \xrightarrow{i} & X \\ r \circ i = I_A & & \searrow & \downarrow \text{r} \\ & & & A \end{array}$$

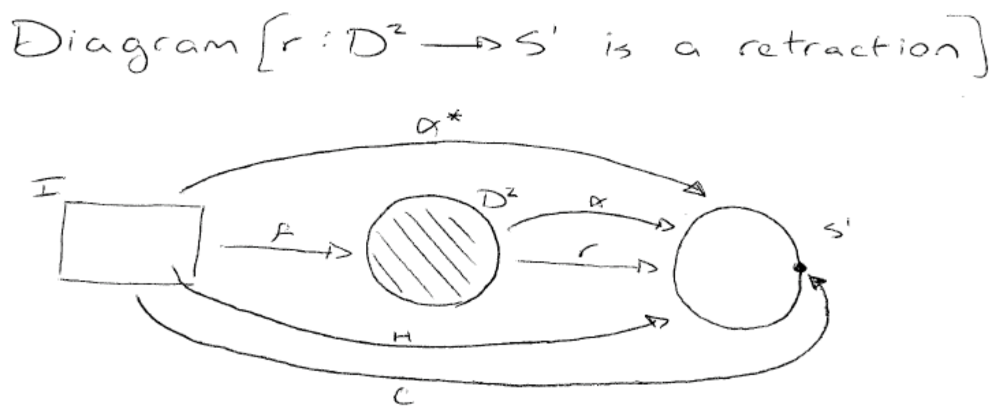
Note: $i : A \to A$ is called the inclusion map. For all $a \in A$, $i(a) = a$.

Corollary - D^2 does not retract onto S^1. Aka - There is no retraction $r : D^2 \to S^2$. Proof through contradiction or though commutative diagrams with fundamental groups.

Commutative Diagram Proof Idea - The problem lies in that $\Pi_1(D^2, (1,0)) \cong \{1\}$. The identity map $Ident_*$ (diagnal line from $\Pi_1(S^1, (1,0))$ to $\Pi_1(S^1, (1,0))$) maps every $\mathbb{Z}$ to itself while $r_* \circ i_*$ only maps to one element in the bottom $\Pi_1(S^1, (1,0))$. Thus, $\Pi_1(S^1, (1,0)) \neq Ident_*$. This idea is based off Prop. 3.24.A which is discussed later.

Proof: Suppose $r : D^2 \to S^1$ is a retraction. From the path lifting lemma proof we know

$$\pi_1(S^1, (1,0)) \cong \mathbb{Z}. \text{ Let } \phi : \pi_1(S^1, (1,0)) \to \mathbb{Z}.$$



The retraction $r : D^2 \to S^1$ gives a map $r/_{S^1} : S^1 \to S^1$ which is the identity map.
Recall homework 5 problem 2, where we have $\alpha : (S^1, (1,0)) \to (X, x_0)$. Let the identity map be this α. Now we have a map $r : D^2 \to S^1$ such that $r/_{\partial D^2} = \alpha$. Therefore there exists $\alpha^* : (I, \partial I) \to S^1$ with $\alpha^* \simeq c$ rel ∂I. Thus,

$$\phi([\alpha^*]) = 1 \text{ because the image of } \alpha^* \text{ wraps 1 time around } S^1.$$

$$\phi([c]) = 0 \text{ because the image of } c \text{ wraps 0 times around } S^1.$$

.

But $\alpha^* \simeq c$ rel ∂I so $[\alpha^*] = [c]$, so this contradicts ϕ being well-defined. Therefore there cannot be a retraction $r : D^2 \to S^1$. $\square$

Theorem : Brower Fixed Point Theorem - Every continuous $f : D^1 \to D^1$ has a fixed point. Use corollary above to prove.

Definition: Induced Homomorphism on Π_1 - Suppose there is a function $f : (X, x_0) \to (Y, y_0)$ and suppose $\alpha : (I, \partial I) \to (X, x_0)$ is a loop in X. Then $f \circ \alpha : (I, \partial I) \to (Y, y_0)$ is a loop in Y. This gives a map $f_* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$ by

$$f_*([\alpha]) = [f \circ \alpha]$$

f_* is call the induced map on Π_1 and is a well-defined homomorphism, which will be proved now.

Proof: : f_* from above is a well-defined homomorphism -

Well-Defined - Let $f : (X, x_0) \to (Y, y_0)$ and let $\alpha : (I, \partial I) \to (X, x_0)$ be a loop in X.
Assume there exists $f_* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$ by $f_*([\alpha]) = [f \circ \alpha]$.
Prove that if $\alpha \simeq \beta$ rel ∂I then $f \circ \alpha \simeq f \circ \beta$ rel ∂I or, If $[\alpha] = [\beta]$ then $f_*([\alpha]) = f_*([\beta])$.

Proof - Let $\alpha, \beta : (I, \partial I) \to (X, x_0)$ be loops in X. Let $[\alpha] = [\beta]$. Finally, let $f : (X, x_0) \to (Y, y_0)$. Then $f \circ \alpha, f \circ \beta : (I, \partial I) \to (Y, y_0)$.
Assume $\alpha \simeq \beta$ rel ∂I so we know there exists a homotopy, say $H(s,t) : (I \times I, \partial I \times I) \to (Y, y_0)$ such that

$$\begin{gathered} H(s,0) = \alpha(s) \\ H(s,1) = \beta(s) \\ H(0,t) = x_0 \\ H(1,t) = x_0. \end{gathered}$$

Define another homotopy $F(s,t) : (I \times I, \partial I \times I) \to (Y, y_0)$ by $F(s,t) = f \circ H(s,t)$. So,

$$\begin{gathered} F(s,0) = f \circ H(s,0) = f \circ \alpha(s) \\ F(s,1) = f \circ H(s,1) = f \circ \beta(s) \\ F(0,t) = f \circ H(0,t) = f \circ x_0 = f(x_0) = y_0 \\ F(1,t) = f \circ H(1,t) = f \circ x_0 = f(x_0) = y_0. \end{gathered}$$

Notice that F is a composition of the continuous function f and the homotopy H, so it is clearly continuous.
Thus, F is a homotopy from $f \circ \alpha$ to $f \circ \beta$ rel ∂I.
So $f \circ \alpha \simeq f \circ \beta$ rel ∂I therefore $[f \circ \alpha] = [f \circ \beta]$ hence $f_*([\alpha]) = f_*([\beta])$ which is what we need to show f_* is well-defined.

Proof of f_* being a homomorphism. Aka - $f_*([\alpha]) \cdot_{\Pi_Y} f_*([\beta]) = f_*([\alpha] \cdot_{\Pi_X} [\beta])$ **for all** $[\alpha], [\beta] \in \Pi_1(X, x_0)$.
Let $[\alpha], [\beta] \in \Pi_1(X, x_0)$ then LHS

$$\begin{aligned} f_*([\alpha]) \cdot f_*([\beta]) &= [f \circ \alpha] \cdot [f \circ \beta] \\ &= [f \circ \alpha \cdot f \circ \beta] \end{aligned}$$

$$\text{where } (f \circ \alpha \cdot f \circ \beta)(s) = \begin{cases} f \circ \alpha(2s) & \text{for } s \in [0, 1/2] \\ f \circ \beta(2s-1) & \text{for } s \in [1/2, 1] \end{cases}$$

Next, RHS

$$\begin{aligned} f_*([\alpha \cdot \beta]) &= f_*([\alpha \circ \beta] \\ &= [f \circ (\alpha \cdot \beta)] \end{aligned}$$

$$\text{where } f \circ \alpha \cdot \beta) = f(\begin{cases} \alpha(2s) & \text{for } s \in [0, 1/2] \\ \beta(2s-1) & \text{for } s \in [1/2, 1] \end{cases})$$

It is clear to see that LHS = RHS. Thus, f_* is a homomorphism.
Now, we have proved that f_* is a well-defined homomorphism. □

Prop. 3.24.A - Fundamental Group Maps(Commutative Diagrams)
1. If there exists situation $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ then $(g \circ f)_* = g_* \circ f_*$.
2. If $I_X : (X, x_0) \to (X, x_0)$ is the identity map the space X then $(I_X)_* : \Pi_1(X, x_0) \to \Pi_1(X, x_0)$ is the identity map on the group $\Pi_1(X, x_0)$.

Prop. 3.24.B - Fundamental Group Maps
If $f \simeq g : (X, x_0) \to (Y, y_0)$ rel x_0 then $f_* = g_* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$. Proved in class - Idea is that we need to show $f_*([\alpha]) = g_*([\alpha])$ by $f_*([\alpha]) = [f \circ \alpha] = [g \circ \alpha] = g_*([\alpha])$. Middle steps were proved in homework 7, problem 4.

Theorem - If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism the $f_* : \Pi(X, x_0) \to \Pi(Y, y_0)$ is an isomorphism. [Proved in class - idea - We've proved f_* is a homeomorphism, just need to show f_* has an inverse]
Sketch proof - Claim g_* is inverse of f_*. So $g_* \circ f_* = (g \circ f)_*$[from prop. 3.24.A.1] $= (I_X)_*$[because $g = f^{-1}$] $=$ identity on $\Pi_1(X, x_0)$[from prop 3.24.A.2] and $f_* \circ g_* = (f \circ g)_* = (I_Y)_* =$ identity on $\Pi_1(Y, y_0)$. So, g_* is the inverse of f_* therefore f_* is invertible and is thus 1 - 1 and onto.

Corollary to prop 3.24.B - If $r : X \to A$ is a strong deformation retraction $[I_X \simeq r]$ then $\Pi(X, x_0) \cong \Pi(A, x_0)$. This corollary was used heavily on homework 9 to find the fundamental groups of weird spaces by sdr-ing them to spaces we know the fundamental group of.
Spaces we know the fundamental group of and used in homework 9 are:
1. $\Pi_1(S^1, x_0) \cong \mathbb{Z}$ furthermore the wedge of two circles $\cong \mathbb{Z} * \mathbb{Z}$ [* is the "free product"]
2. $\Pi_1(S^2, x_0) \cong \{1\}$ furthermore the wedge of two spheres $\cong \{1\} * \{1\} \cong \{1\}$ [* is the "free product"]
3. $\Pi_1(\mathbb{R}P^2, y_0) \cong \Pi_1(X, x_0) \cong \mathbb{Z}_2$
4. [Annulus: $S^1 \times I$] - $\Pi_1(S^1 \times I, (x_0.y_0)) \cong \Pi_1(S^1, x_0) \times \Pi_1(I, y_0) \cong \mathbb{Z} \times \{1\} \cong \mathbb{Z}$
5. $\Pi_1(T^2 - pt, (x_0, y_0)) \cong \Pi_1(S^1, x_0) \times \Pi_1(S^1, x_0)$ [wedge of 2 circles] $\cong \mathbb{Z} * \mathbb{Z} \cong \langle a, b : ab\bar{a}\bar{b} \rangle$
6. [Klein Bottle] - $\Pi_1(KB, x_0) = \langle a, b : ba\bar{b}a \rangle$

Proof: Sketch Proof - Suppose r is an sdr. Let $i : A \to X$ be the inclusion map. So $r \circ i = I_A$ and from prop 3.24.A.1 $r_* \circ i_* = (I_A)_*$. Since r is sdr $r \simeq I_x$ and from prop 3.24.B $i \circ$

$r \simeq i \circ I_X$ and $i_* \circ r_* = i_* \circ (I_X)_*$ so from 3.24.A.2$r_* : \Pi(X, x_0) \to \Pi(A, x_0)$. Thus from prop 3.24.A r_* is a homomorphism because it is the identity map and thus $\Pi(X, x_0) \cong \Pi(A, x_0)$. $\square$

Fact - Group Theory - 2 groups with operations $(G, \cdot_G)$ and $(H, \cdot_H)$ can define a new group $G \times H$ as follows: Note, the elements of $G \times H$ are ordered pairs (g, h). Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$ then to multiply two elements of $G \times H$:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H h_2)$$

where $g_1 \cdot_G g_2$ is an element of G and $h_1 \cdot_H h_2$ is an element of H.

Theorem - Let $x_0 \in X, y_0 \in Y$. Then $\Pi_1(X \times Y, (x_0, y_0)) \cong \Pi_1(X, x_0) \times \Pi_1(Y, y_0)$.

Free Groups - Word Mult. - $ab \cdot ab = abab$. [Mult. reduced words, then cancel] This is an example of a free group rank 2[F_2] - element is made up of a and b. Presentation of $F_2 = \langle a, b : \rangle$

Definition: Finite Presentation for a group G - is $\langle g_1, g_2, ..., g_n : R_1, R_2, ..., R_n \rangle$ [g_i's are generators] so that
1. Every element of G is a finite product of g_i's and their inverses.
2. Each relation R_j ia a word in the g_i's and their inverses which gives the identity element in G.
3. If a reduced word in g_i's in the identity element in G, then that reduced word is obtained by reducing some product of **conjugates** of the relations.

Conjugates - A conjugate of a word $w \in$ G is gwg^{-1}, where g is an element on G.
Example - Conjugate of the relation $ab\bar{a}\bar{b}$ is

$$(a\bar{b}\bar{a}b)(ab\bar{a}\bar{b})(\bar{b}ab\bar{a})$$

Notice the center word contracts to the empty word and the outer words are inverses.

Theorem - Every finite group have a finite presentation.

Example - $G = \{e, a, b\}$ with e being the identity element, so, from definition of a group a and b must be inverses.

[make table with every generator on axis, perform multiplication to get interior words, (top generator $\cdot$ side generator)]

$G \cong \langle e, a, b : ab, ba, ee, aa\bar{b}, \text{etc} \rangle$. The 2 element words are one that become e [the identity element] on the table [these elements are inverses]. The three element words, the third element is the inverse of the first two multiplied together.

Definition - The free product of two finitely presented groups G and H is G * H = ¡generatiors for G, generators for H: relations for G, relation for H¿.

Definition: Abelian - A group is abelian if $ab = ba$ for all $a, b \in G$.

Definition: Disjoint Union - $A \sqcup B$ - If A and B both contain 2 then $A \sqcup B$ will contain 2_A and 2_B

Definition: Attaching a Disk to Y using F - $X = Y \sqcup D^2 / \sim = Y \cup_f D^2$ - Like patching the whole in the space. Allows us to use group presentation for spaces like the torus and the klein bottle. Attaching a disk using f has the effect of killing the loop in Y represented by the boundary of the disk [makes it equivalent to the empty word - the identity]. We can imagine stretching the disk and manipulating it into the shape of a know loop in the space, "killing" that loop.

Example - Let $f : \partial D^2 \to Y$ and let $Y = S^1$, then $X = S^1 \sqcup D^2$.
We know $\Pi_1(Y, y_0) \cong \mathbb{Z} \cong \langle a : \rangle$ but
$\Pi_1(X, y_)) \cong \langle a : a \rangle$ [because f "kills" loop a] $\cong \{1\}$.

Definition: **Closed** Orientable Surfaces - Classification Theorem - Every closed,orientable surface is homeomorphic to a surface of genus g, for some non-negative integer g. S_g = closed orientable surfce of Genus g (g is the number of holes on the surface: $S^2 = S_0$ and $T^2 = S_1$).

S_2 is the two holed torus [becomes and octagon] and $\Pi_1(S_2, *) \cong \langle a, b, c, d : \bar{a}ba\bar{b}c\bar{d}\bar{c}d \rangle$.
S_3 is the two holed torus $\Pi_1(S_3, *) \cong \langle a, b, c, d, e, f : ab\bar{a}\bar{b}cd\bar{c}\bar{d}ef\bar{e}\bar{f} \rangle$.
Closed - A topological space is closed if it has no boundary and has a finite diameter.

Definition: Knot Complement - Let k be a knot: $k = f(s)$. Then $X = S^3 - k$ is the knot compliment of k.

Definition - Let $f : S^1 \to S^3$ and let $k = f(s')$ be the knot then $\Pi_1(k, *) \equiv \Pi_1(S^3 - k, *)$.
The fundamental group of the knot is $\Pi_1(S^3 - f(s'), *)$

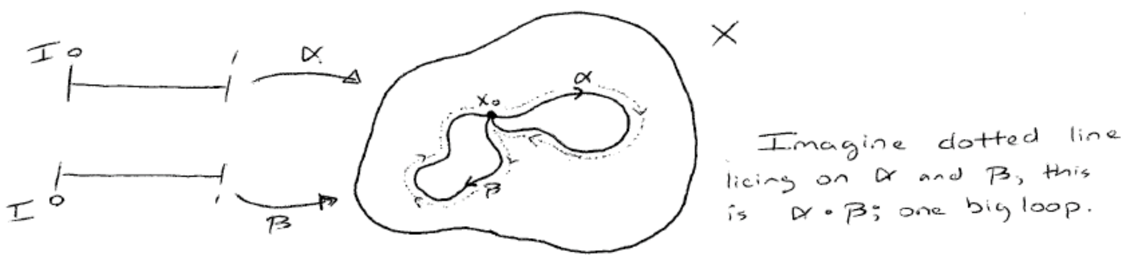
Finding relations of a knot - Orient the overpass so it travels left to right and underpass goes down to up. Using right hand rule and imagination find the relations. Pattern - overpass $cdot$ (top of underpass)$^{-1}$ = (bottom of underpass)$^(-1)$ $\cdot$ overpass.

Definition: Covering Space Projection - $p : E \to B$ is a covering space projection if E and B are both connected and for all $b \in B$ there exists a path connected **neighborhood** U of b such that every component of $p^{-1}(U)$ maps homeopathically (via p) onto U.

Neighborhood - A neighborhood of b is an open set containing b. **Degree -** The degree of a covering space is the number of pre-images of any point $b \in B$.

Definition: Universal Cover - A simply connected cover. Aka - A cover with the trivial fundamental group.

Known Universal covers
Infinite tree - universasla cover of the wedge of two circles.
$\mathbb{R}^1$ - universal cover of S^1.
$\mathbb{R}^2$ - universal cover of T^2.



Definition: Regarding Covering Spaces -
1. Homeo(X) = $\{h : X \to X : h$ is a homeomorphism $\}$
2. Let G be a group and X be topological space. We say that G acts on X if there exists a homomorphism

$$\text{T: G} \to \text{Homeo(X)}$$

Intuitively, a group action ia a way to think of each group element as a homeomorphism from X to itself. T [Tau] is a translation of E such that $T : E \to E$.
3. A group action is **free** if $u \neq 1 \Rightarrow T_u$ has no fixed points.
4. Let $T : G \to$ Homeo(X) be a group action. T is faithful if it is 1 - 1.
5. H is a subset of G if $H \subseteq G$ and H and G are groups with some operation.

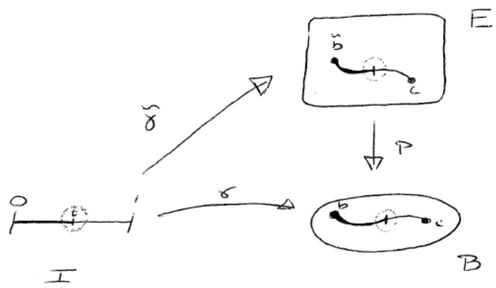
Definition: Construction of Quotient Spaces - Let H ¡ G, E a topological space, and $G = \Pi(E, *)$. Define E/H, the quotient of E by H to be

$$E/H = E/x \sim T_h(x) \text{ for } x \in E, h \in H$$

Definition: Unique Path Lifting Property - If $p : E \to B$ is a covering space projection and $\gamma : (I, 1, 0) \to (B, b, c)$ is a path in B. Choose $\widetilde{b} \in p^{-1}(b)$.
Then there exists a unique path $\widetilde{\alpha} : (I, 0) \to (E, \widetilde{b})$ such that $p \circ \widetilde{\alpha} = \alpha$.
Proved uniqueness and existence. Idea behind existence [like Lemma] - need to use local homeomorophism. We can lift $\gamma/_{[0,t]}$ where t is somewhere between 0 and 1, hence between $\widetilde{b}$ and $\widetilde{c}$ and between b and c. We know at t there exists an open neighborhood surrounding is and in that neighborhood is a part of $\widetilde{\alpha}$ we didn't have before. By moving t to the part of $\widetilde{\alpha}$ on the boundary of the neighborhood [and doing this over and over again], we will move t closer to c along α, we will eventually get us to c. Making $p \circ \widetilde{\alpha} = \alpha$. See diagram on back page.

Definition: Facts about Quotient Spaces -
1. If $p : E \to B$ is a covering, then subgroups of $\Pi_1(B, b_0)$ yield unique covering spaces of B.

Unique Path Lifting Property Diagram



2. If $p : E \to B$ is a covering then p gives rise to a unique subgroup of $\Pi_1(B, b_0)$.
Key Point - There is 1 - 1 correspondence between subgroups of $\Pi_1(B, b_0)$ and cover $p : E \to B$(mod covering space equivalence).

Theorem : Let X is simply connected and locally path connected, $p : E \to B$ a covering, and $f : X \to B$ be continuous. Then there exists a continuous lift $\widetilde{f} : X \to E$ so that $p \circ \widetilde{f} = f$. Furthermore $\widetilde{f}$ are in 1 - 1 correspondence with the primage of $f(x_0)$..

Main Idea to proof - Choose a fixed point $e_0 \in p^{-1}(b_0)$ and define $\widetilde{f} = e_0$. Let $y \in X$ and γ be a path from x to y. Lift γ to $\widetilde{\gamma}$ starting at e_0. Define $\widetilde{f}(y) = \widetilde{\gamma}(1)$. Key fact is that X is simply connected so it insures that had we chosen a different path from x to y, we would still end up with the same $\widetilde{f}$.

Definition: Covering Transformation - Let $p : \widetilde{X} \to X$ be a covering, then the covering transformation ia a map $\gamma : \widetilde{X} \to \widetilde{X}$ so that there is a commutative diagram between $\widetilde{X}, \widetilde{X}$, and X.