Definitions and Examples for Topology

Abelian
A group is *abelian* if $ab = ba$ for all $a, b \in G$.

Bijection
A *bijection* from X to Y is a map $f : X \longrightarrow Y$ which is both one-to-one and onto.

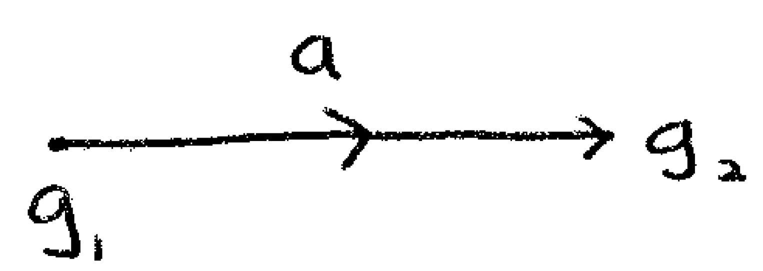
Brouwer Fixed Point Theorem
Every continuous $f : D^2 \longrightarrow D^2$ has a fixed point.

Cayley Graph
The *Cayley Graph* of a (finitely generated) group is a graph consisting of one vertex for each group element, and out of each vertex, one directed edge per generator, so that if

$$g_1 \cdot a = g_2$$

↑ generator

add edge from



Conjugate
A *conjugate* of a word w in G is gwg^{-1}, where g is an element of G.

Example
$\mathbb{Z} \times \mathbb{Z} \simeq < a, b | aba^{-1}b^{-1} >$
$\phi : < a, b | > \longrightarrow \mathbb{Z} \times \mathbb{Z}$ by
$\phi(a) = (1, 0)$
$\phi(b) = (0, 1)$
$\phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)\phi(a^{-1})\phi(b^{-1}) = (1,0) + (0,1) + (-1,0) + (0,-1) = (0,0).$

Constant
A function $f : X \longrightarrow Y$ is *constant* if there is some $c \in Y$ so that

$$f(x) = c$$

for every $x \in X$.

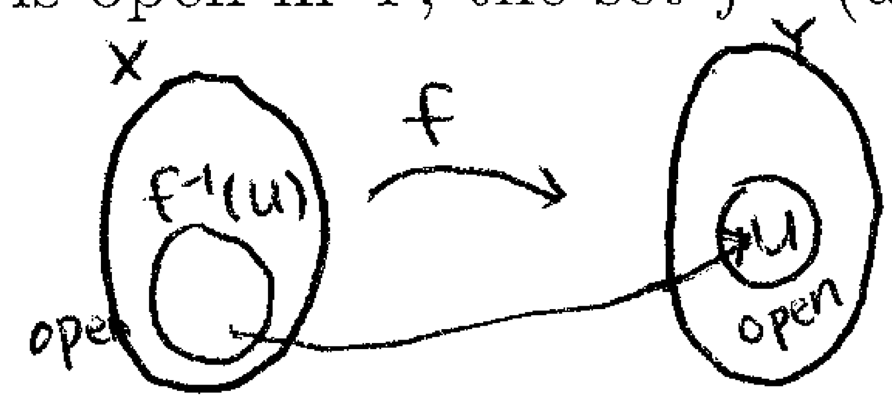
Constant loop
The *constant loop* in a space X based at $x_0 \in X$ is

$$C : (I, dI) \longrightarrow (x, x_0)$$

by $C(s) = x_0$.

Continuity

A map f from a topological space X to a topological space Y is *continuous* if and only if for every set $U \subseteq Y$ that is open in Y, the set $f^{-1}(u) \subseteq X$ is open in X.



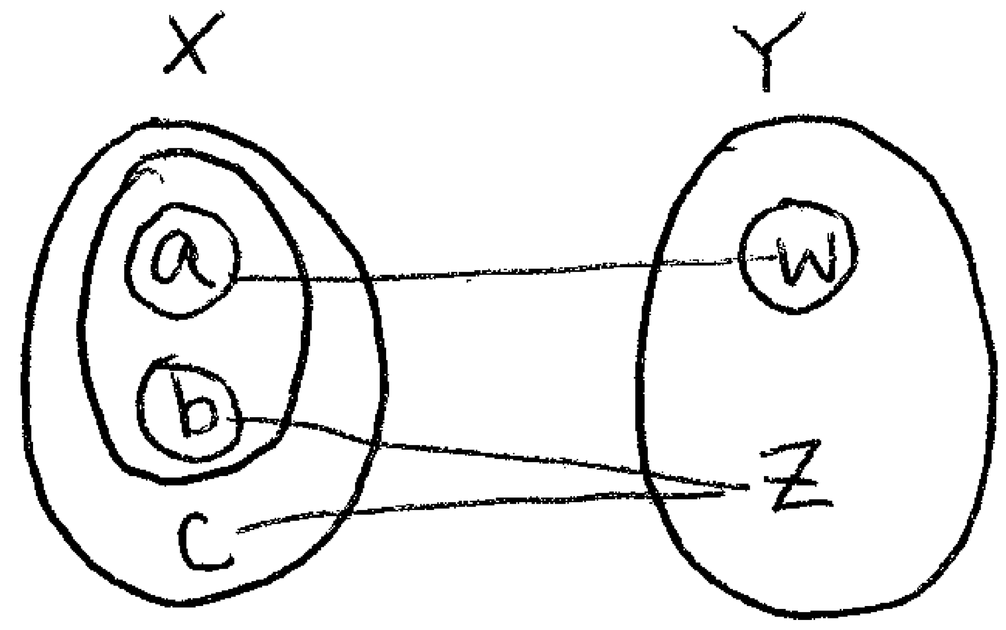
Example

X = {a, b, c} $\mathcal{T}_x = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$

Y = {w, z} $\mathcal{T}_x = \{\emptyset, Y, \{w\}\}$

Define $f : X \longrightarrow Y$ by

$$f(a) = w$$
$$f(b) = z$$
$$f(c) = z$$



Is f continuous?

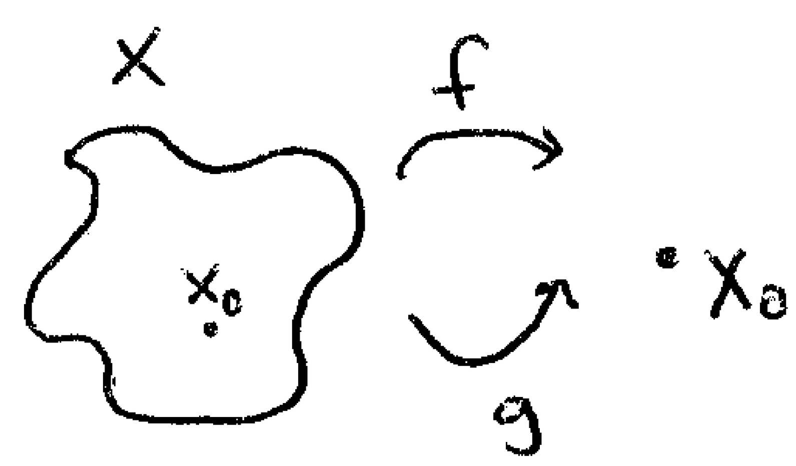
The open sets in Y are $\emptyset, Y, \{w\}$.

$$f^{-1}(\emptyset) = \emptyset \in \mathcal{T}_x$$
$$f^{-1}(Y) = X \in \mathcal{T}_x$$
$$f^{-1}(\{w\}) = \{a\} \in \mathcal{T}_x$$

So f is continuous.

Contractible

A space which is homotopy equivalent to a point is called *contractible.*



Proposition

X is contractible if and only if $I_x \simeq$ constant : $X \longrightarrow X$.

Proof

=> Assume X is contractible. We want to show $I_x \simeq$ constant. Then by definition, X is homotopy equivalent to a point. So there's some point y and maps

$$f : X \longrightarrow \{y\}$$
$$g : \{y\} \longrightarrow X$$

so that

$$g \circ f \simeq I_x$$
$$f \circ g \simeq I_{\{y\}}$$

Note, if $x \in X$, then $g \circ f(x) = g(y)$.

So, $g \circ f$ maps every point in X to the same point in X (namely $g(y)$). So $g \circ f$ is a constant map. So $I_x \simeq$ constant map.

<= Assume $I_x \simeq$ constant: $X \longrightarrow X$. Then $I_x \simeq C$, where C is a map $C : X \longrightarrow X$ and $x_0 \in X$ and $C(X) = x_0$ for all $x \in X$. We want to show X is contractible.

Define $f : X \longrightarrow \{x_0\}$ by $f(x) = x_0$ for all $x \in X$.

Define $g : \{x_0\} \longrightarrow X$ by $g(x_0) = x_0$.

Now $f \circ g(x_0) = f(x_0) = x_0$.

So $f \circ g = I_{\{x_0\}}$.

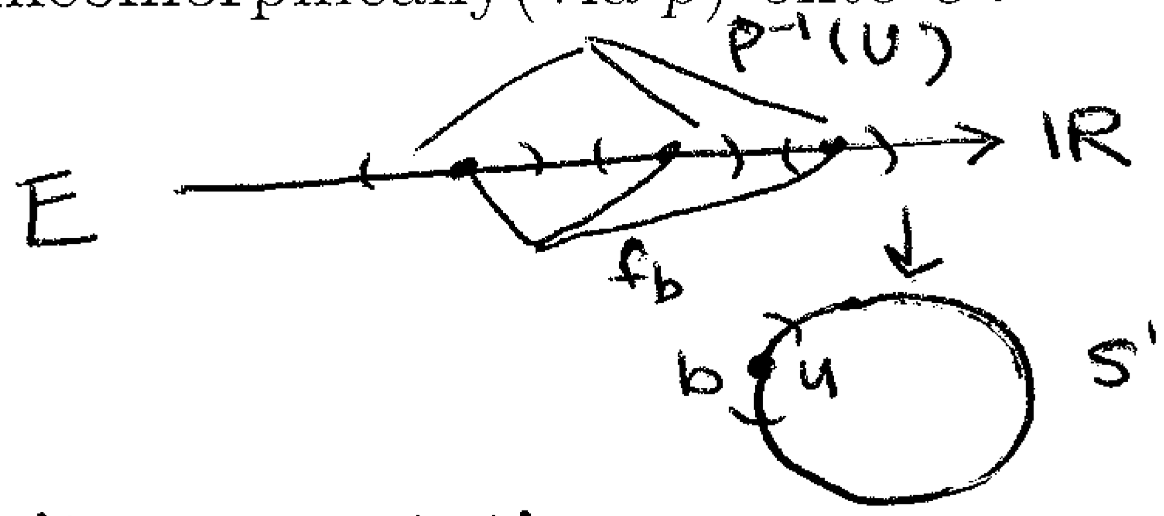
Therefore, $f \circ g \simeq I_{\{x_0\}}$.

Further, $g \circ f(x) = g(x_0) = x_0$.

So $g \circ f = C \simeq \mathcal{T}_x$.

So X is homotopy equivalent to a point; X is contractible.

Covering space
$p : E \longrightarrow B$ is a *covering space*(projection) if E and B are path connected and *for all $b \in B$, there exists a path connected neighborhood U of b such that every component of $p^{-1}(u)$ maps homeomorphically(via p) onto U.



Finite presentation
A *finite presentation* for a group G is $< g_1, g_2, g_3, ..., g_n | R_1, R_2, ..., R_m >$ so that

1. Every element of G is a finite product of g_i's and then inverses. (g_i]s are generators).
2. Each relation R_j is a word in the g_i's (and their inverses) which gives the identity element in G.
3. If a reduced word in g_i's is the identity element in G, then that reduced word is obtained by reducing some product of conjugates of the relatives.

Fixed point
A *fixed point* of a function $f : X \longrightarrow x$ is an element $x \in X$ so that $f(x) = X$.

<u>Example</u>
$f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = 2x$.
0 is the only fixed point.

Free product
The *free prudct* of two finitely presented groups G and H is
$G * H =$ ¡generators for G, generator for H relations for G, relations for H¿

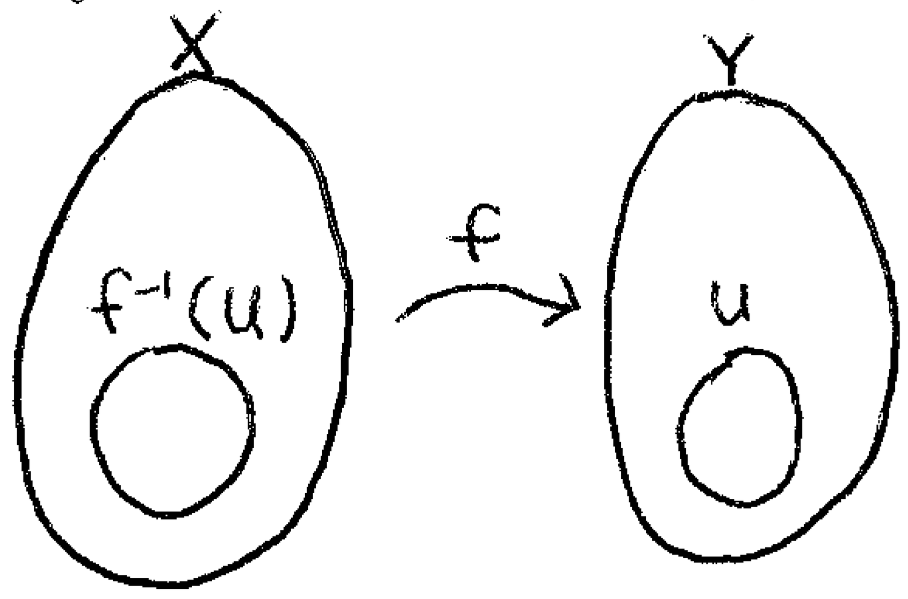
<u>Example</u>

$$\begin{aligned} \mathbb{Z} &\simeq < || > \\ &= < a| > \\ &= < b| > \end{aligned}$$

$\mathbb{Z} * \mathbb{Z} \simeq < a, b| >= F_2$
$\mathbb{Z} \times \mathbb{Z} \simeq (a, b)|aba^{-1}b^{-1}$

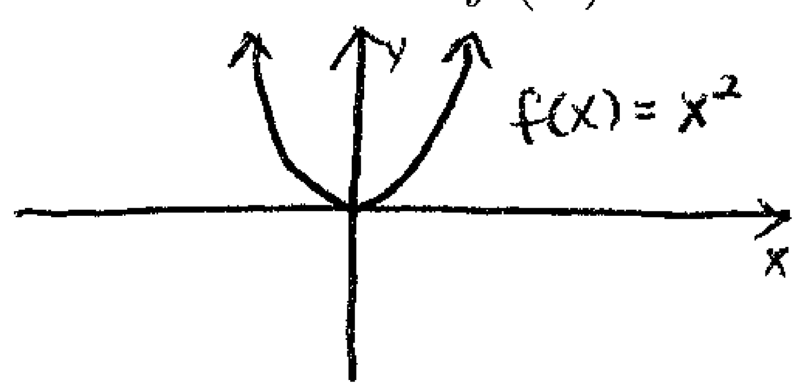
Function

If $f : X \longrightarrow Y$ is a *function* and $U \subseteq Y$, then $f^{-1}(u) = \{x \in X : f(x) \in U\}$.



Example

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is $f(x) = x^2$



$f^{-1}(\{2\}) = \{\sqrt{2}, -\sqrt{2}\}$
$f^{-1}(\{-2\}) = \emptyset$
$f^{-1}((1,2)) = (-\sqrt{2}, -1)U(1, \sqrt{2})$

Group

A *Group* consists of a set G and an operation $\cdot$ defined on that set $(\cdot : G \times G \longrightarrow G)$ so that

1. Identity: There is an element $e \in G$ so that for every $g \in G$

$$e \cdot g = g$$
$$g \cdot e = g$$

2. Inverses: For all $g \in G$, there is $h \in G$ so that

$$g \cdot h = e$$
$$h \cdot g = e$$

3. Associativity: If $g, h, j \in G$, then

$$(g \cdot h) \cdot j = g \cdot (h \cdot j)$$

Homeomorphic

Two topological spaces X and Y are *homeomorphic* if there exists a homeomorphism $F : X \longrightarrow Y$.

Homeomorphism

A bijection $f : X \longrightarrow Y$ such that f and f^{-1} are both continuous is called a *homeomorphism.*

Homeotopic

Two maps $f, g : X \to Y$ are *homeotopic* if there exists a map

$$F : X \times I \longrightarrow Y$$

such that for each $x \in X$

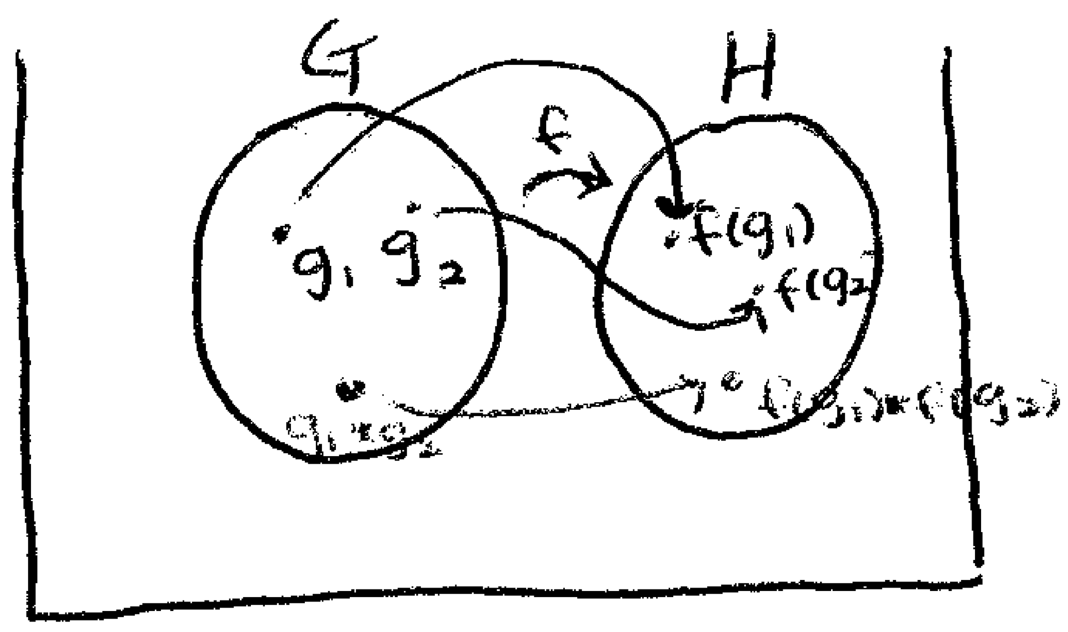
$$F(x, 0) = f(x) \text{ and}$$
$$F(x, 1) = g(x)$$

We say f is homotopic to g, write $f \simeq g$, and F is a *homotopy* between f and g.

Homomorphic

Two groups $(G, *_G)$ and $(H, *_H)$ are *homomorphic* if there exists a function $f : G \longrightarrow H$ so that, for every $g_1, g_2 \in G$,

$$f(g_1 *_G g_2) = f(g_1) *_H f(g_2)$$



Homotopic

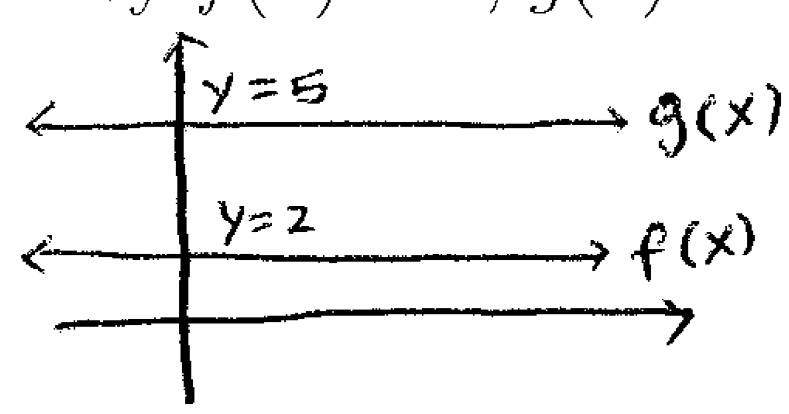
Two paths $f, g : I \longrightarrow X$ are *homotpic* if and only if there exists

$$F : I \times I \longrightarrow X$$

so that

$$F(x, 0) = f(x)$$
$$F(x, 1) = g(x)$$

<u>Example</u>
Define $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = 2$, $g(x) = 5$, for every $x \in /R$.



Show f and g are homotopic. We need a map

$$F : \mathbb{R} \times I \longrightarrow \mathbb{R}$$

so that

$$F(x, 0) = f(x) = 2$$
$$F(x, 1) = g(x) = 5$$

Define $F : \mathbb{R} \times I \longrightarrow \mathbb{R}$ by

$$F(x, t) = 3t + 2$$

So

$$F(x, 0) = 3(0) + 2 = 2 = f(x)$$
$$F(x, 1) = 3(1) + 2 = 5 = g(x)$$

Homotopic relative to A

Two maps $f, g : (X, A) \longrightarrow (Y, B)$ are *homotopic relative to* A, written $f \simeq grelA$, if there exists a homotopy $F : (X \times I, A \times I) \longrightarrow (Y, B)$so that

$$F(x, 0) = f(x),\ F(x, 1) = g(x) \text{ for all } x \in X \text{ and for all } x \in A$$
$$F(x, t) = f(x) \text{ for every } t \in I.$$

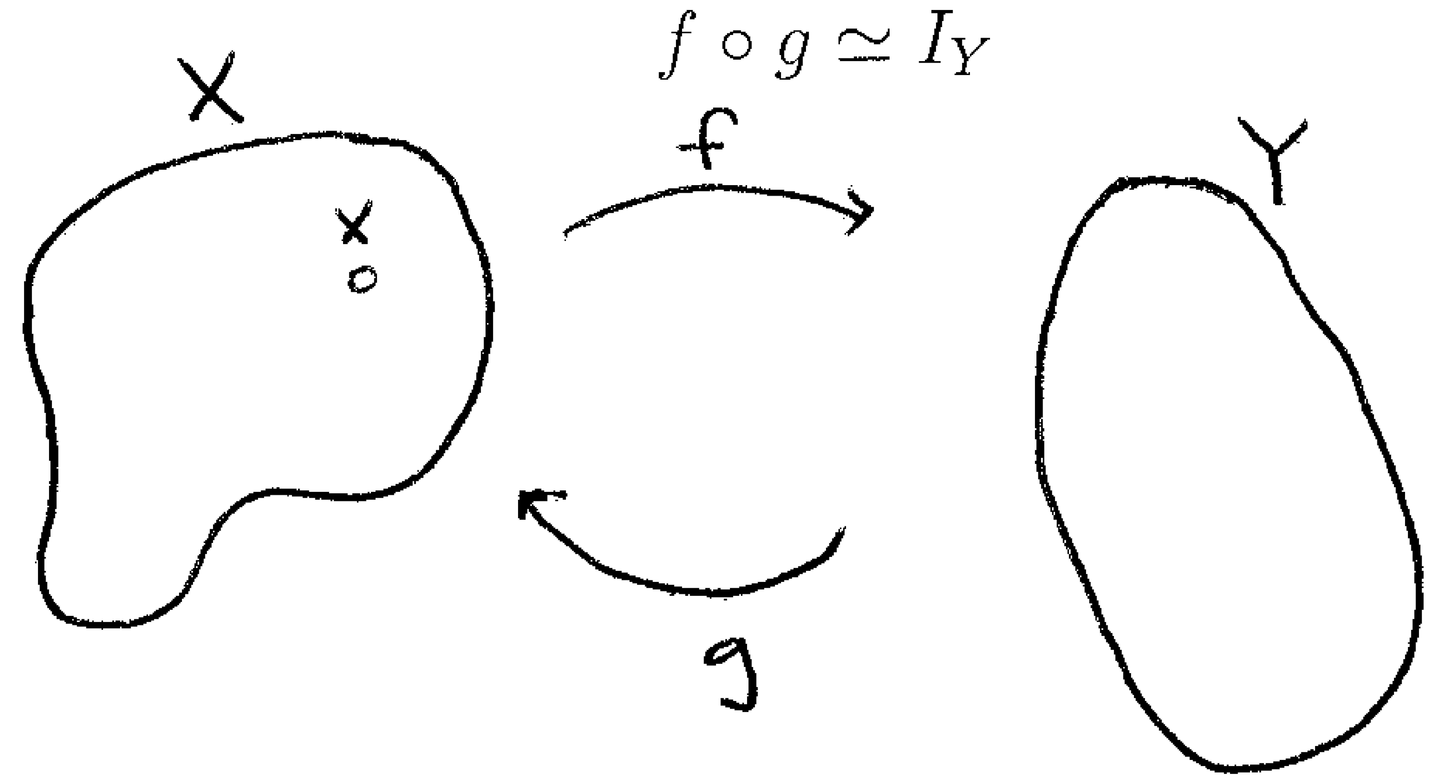
Homotopy class

The *homotopy class* of a loop α in X based at $x_0 \in X$ is

$$[\alpha] = \{\beta : (I, dI) \longrightarrow (x, x_0) : \alpha \simeq \beta \text{ rel dI } \}$$

Homotopy equivalent
Two spaces X and Y are *homotopy equivalent* if there exists maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ so that

$$g \circ f \simeq I_x$$
$$f \circ g \simeq I_Y$$



Identity map
Let X be a topology space. Then I_x is the function $I_x : X \longrightarrow X$ defined by

$$I_x(x) = X$$

This is called the *identity map* on X.

Inclusion map
If $A \subseteq X$, then the *inclusion map* $i : A \longrightarrow X$ is $i(a) = a$ for all $a \in A$.

Induced map
Suppose $f : (X, x_0) \longrightarrow (Y, y_0)$ and suppose $\alpha : (I, dI) \longrightarrow (X, x_0)$ is a loop in X. Then $f \circ \alpha : (I, dI) \longrightarrow (Y, y_0)$ is a loop in Y. This gives a map $f_x : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$,

$$f_*([\alpha]) = [f \circ \alpha]$$

f_* as above is called the *induced map* on π_1.

Proposition: f_* is a homomorphism
Proof: We want to show $f_*([\alpha]) \cdot f_*([\beta]) = f_*([\alpha] \cdot [\beta])$ for all $[\alpha][\beta] \in \pi_1(X, x_0)$. Let $[\alpha][\beta] \in \pi_1(X, x_0)$.
Then, $f_*([\alpha]) \cdot f_*([\beta]) = [f \circ \alpha] \cdot [f \circ \beta] = [f \circ \alpha \cdot f \circ \beta]$ where

$$(f \circ \alpha \cdot f \circ \beta)(s) = \begin{cases} f \circ \alpha(2s) & 0 \leq s \leq 1/2 \\ f \circ \beta(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

$$f_*([\alpha] \cdot [\beta]) = f_*([\alpha \cdot \beta]) = [f \circ (\alpha \cdot \beta)]$$

where,

$$(f \circ (\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ \beta(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

These are the same.

Isomorphic
Two groups $(G, *_G), (H, *_H)$ are *isomorphic*,if there exists $f : G \longrightarrow H$ which is a bijective homomorphism. $G \simeq H$.

Proposition
$(G, \cdot_G) \simeq (H, \cdot_H)$

Define $f : G \longrightarrow H$ by
$$f(e) = 1$$
f is clearly a bijection. To see f if a homomorphism, choose $g_1, g_2 \in G$. Then $g_1 = e = g_2$. So,

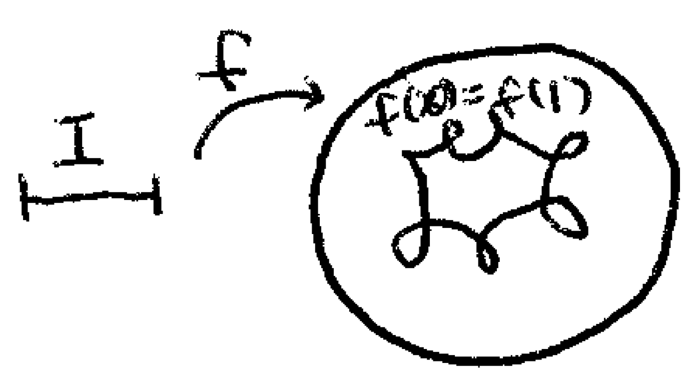
$$\begin{aligned} f(g_1 \cdot_G g_2) &= f(e \cdot_G e) \\ &= f(e) \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(g_1 \cdot_H g_2) &= f(e \cdot_H e) \\ &= 1 \cdot_H 1 \\ &= 1 \end{aligned}$$

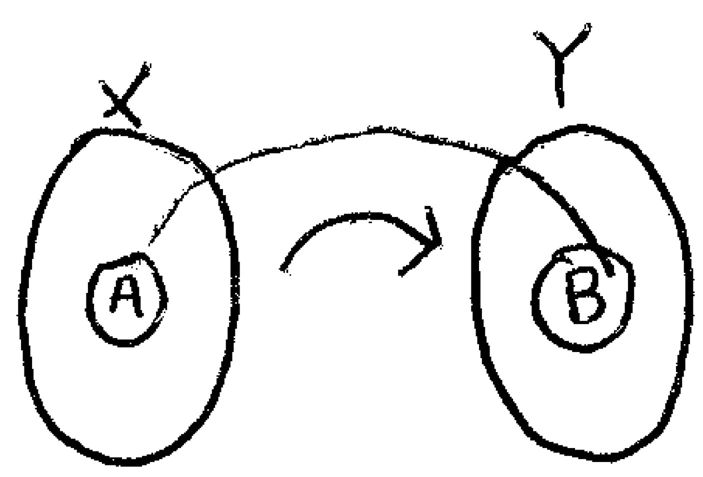
So $G \simeq H$.

Knot complement
Let K be a knot: $K = f(s^1)$. Then $X = S^3 - K$ is the *knot complement* of K.

Loop
A *loop* in X is a path $f : I \longrightarrow X$ so that $f(0) = f(1)$.



Map of Pairs
A *map of pairs* is $f : (x, A) \longrightarrow (Y, B)$ which means
1. $f : X \longrightarrow Y$
2. $f(A) \subseteq B$



Nulhomotopic
A function $f : X \longrightarrow Y$ is *nulhomotpic* if it is homotpic to a constant map.

One-to-One
A function $f : X \longrightarrow Y$ is *one-to-one* if and only if, for all $a, b \in X, f(a) = f(b) \longrightarrow a = b$.

Onto

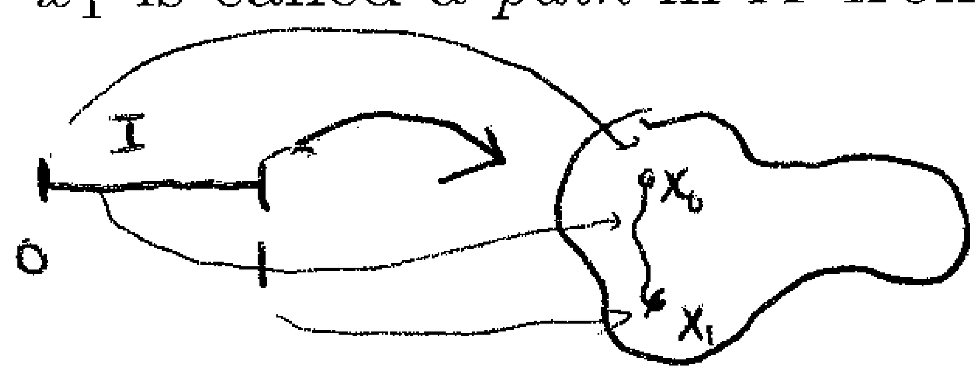
A function $f : X \longrightarrow Y$ is *onto* if and only if, for all $a \in X$, there exists $b \in X$, $f(b) = a$.

Pair

A *pair* of topological spaces (x, A) is a topological space X with $A \subseteq X$.

Path

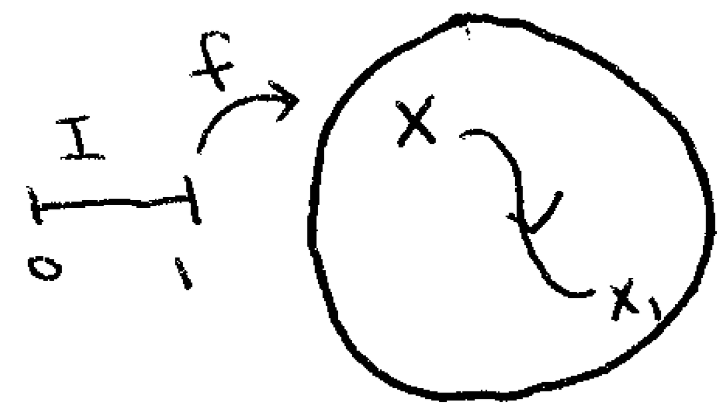
A map $f : I \longrightarrow X$ so that $f(0) = x_0$ and $f(1) = x_1$ is called a *path* in X from x_0 to x_1. (Note: Path is a map, not the image of a map.)



Path Connected

A space X is *path connected* if for every x_0, $x_1 \in X$, there exists a path $f : I \longrightarrow X$ so that

$$f(0) = x_0 \text{ and}$$
$$f(1) = x_1$$



Path Homotopic

Two paths $f, g : I \longrightarrow X$ are *path homotpic* or *homotopic rel endpoints*, or *homotpic rel boundary.* If they have the same initial point x_0 and the same final point x, and there's a homotopy

$$F : I \times I \longrightarrow X$$

so that

$$F(s, 0) = f(s),\ F(s, 1) = g(s) \text{ for all } s \in I$$
$$F(0, t) = x_0,\ F(1, t) = x, \text{ for all } t \in I.$$

Reduced word

A *reduced word* is a word in which you do not have a letter next to its inverse.

Example
$abaa^{-1}b$ Not reduced
abb is reduced.
This group above is the free group of rank 2, written F_2.

Retraction of X onto A

If $A \subseteq X$, then a *retraction of* X *onto* A is a continuous function $r : X \longrightarrow A$ with $r|_A = I_A$ (alternately $r \circ i = I_A$) "Elements of A are fixed by the function r."

Example
Let $A = 0$, $X = \mathbb{R}$, then $A \subseteq X$.
Define $r : X \longrightarrow A$

$$\mathbb{R} \longrightarrow \{0\} \text{ by}$$
$$r(x) = 0.$$

Then $r(0) = 0$, so r fixes elements of A. So r is a retraction.

Simply connected

A space X is called *simply connected* if it is path connected and $\pi_1(X, x_0) \simeq \{1\}$, for every $x_0 \in X$.

Theorem $\pi_1(s^1, x_0) \cong (\mathbb{Z}, +)$

Proof

We want do define a map

$$f : \pi_1(s^1, (1,0)) \longrightarrow \mathbb{Z}$$

which is an isomorphism.

To define f, let $[\alpha] \in \pi_1(s^1, (1,0))$

Then $\alpha : (I, dI) \longrightarrow (s^1, (1,0))$. Using the Unique Path Lifting lemma there exists $\alpha(0) = 0$.

Define

$$n_\alpha = \tilde{\alpha}(1)$$

.

Claim: Note that n_α is an integer.

Proof Claim: we know $\pi \circ \alpha = \alpha$

$$=> \pi(\tilde{\alpha}(1)) = \alpha(1)$$
$$=> \pi(\tilde{\alpha}(1)) = (1,0)$$

The function π is defined by

$$\pi(x) = (cos2\pi x, sin2\pi x) = (1,0)$$

when $cos2\pi x = 1$ and $sin2\pi x = 0$.

This happens only when $x \in \mathbb{Z}$.

So

$$\pi(\tilde{\alpha}(1)) = (1,0) => n_\alpha = \tilde{\alpha}(1) \in \mathbb{Z}$$

.

For $[\alpha] \in \pi_1(s^1, (1,0))$, define $f[\alpha] = n_\alpha = \alpha(1)$.

Need to check f is 1.well-defined 2. 1-1 3.onto 4.homomorphism.

1.Well-defined: Show $[\alpha] = [\beta]$, then $f[\alpha] = f[\beta]$.

Assume $[\alpha] = [\beta]$. In other words, assume $\alpha, \beta : (I, dI) \longrightarrow (s^1, (1,0))$with $\alpha \simeq \beta$ rel dI.

[want to show $n_\alpha = n_\beta$].

Since $\alpha \simeq \beta$ rel dI, there exists homotopy $F : (I \times I, dI \times I) \longrightarrow (s^1, (1,0))$.

So that

$$F(s,0) = \alpha(s)$$
$$F(s,1) = \beta(s)$$
$$F(0,t) = (1,0)$$
$$F(1,t) = (1,0)$$

For each t, let $\alpha_t(s) = F(s,t)$. This is a loop in s^1 based at (1,0). Define function $n : \{\text{loops in } s^1\} \longrightarrow \mathbb{Z}$ by $n(\gamma) = nr$. Since this is a continuous map into a discrete set, it must be constant. So $n_{\alpha_{t_1}} = n_{\alpha_{t_2}}$ for every t_1, t_2.So $n_\alpha = n_\beta$ and f is well defined.

2. 1-1 Suppose $f([\alpha]) = f([\beta])$. We want to show $[\alpha] = [\beta]$.

So $n_\alpha = n_\beta$. We want to define a homotopy $F : (I \times I, dI) \longrightarrow (s^1, (1, o))$. So that

$$f(s,0) = \alpha(s)$$
$$f(s,1) = \beta(s)$$
$$F(0, \text{ or } 1, t) = (1,0)$$

.

Since $\alpha, \beta : (I, dI) \longrightarrow (s^1, (1,0))$, the unique path lifting lemma says there exists lifts $\tilde{\alpha}, \tilde{\beta} : (I, dI) \longrightarrow \mathbb{R}$ with $\tilde{\alpha} = 0 = \tilde{\beta}(0)$.
Since $f[\alpha] = f([\beta])$, we know $\tilde{\alpha}(1) = \tilde{\beta}(1)$.
Define homotopy $F : I \times I \longrightarrow \mathbb{R}$ by

$$\tilde{F}(s,t) = (1-t)\alpha(s) + t(p) \in \mathbb{R}$$
$$\tilde{F}(s,0) = \alpha(s)$$
$$\tilde{F}(s,1) = \beta(s)$$
$$\tilde{F}(0,t) = (1-t)\alpha(1) - t\beta(1) = (-t)\alpha(1) + t\alpha(1) = \alpha(1)$$
$$\tilde{F}(0,t) = (1 - t(0 + t(0) = 0$$

So $\tilde{\alpha} \simeq \tilde{\beta}$ rel dI.
Now define F by $F = \pi \circ \tilde{F}$. Then $F : I \times I \longrightarrow s^1$, and

$$\begin{aligned} f(s,0) &= \pi \circ \tilde{F}(s,0) \\ &= \pi \circ \tilde{\alpha}(s) \\ &= \alpha(s) \end{aligned}$$

$$\begin{aligned} f(s,1) &= \pi \circ \tilde{F}(s,1) \\ &= \pi \circ \tilde{\beta}(s) \\ &= \beta(s) \end{aligned}$$

$$\begin{aligned} f(0,t) &= \pi \circ \tilde{F}(0,t) \\ &= \pi(0) \\ &= (1,0) \end{aligned}$$

$$\begin{aligned} f(1,t) &= \pi \circ \tilde{F}(1,t) \\ &= \pi(\tilde{\alpha}(1)) \\ &= (1,0) \end{aligned}$$

3. <u>onto</u>: Let $m \in \mathbb{Z}$. We want to find $[\alpha]$ such that $f([\alpha]) = m$.
Define $\alpha_m : (I, dI) \longrightarrow (s^1, (1,0))$ by

$$\alpha_m(s) = (\cos(2\pi sm), \sin(2\pi sm))$$

.

Then $\alpha(1) = m$. So $f([\alpha]) = n_\alpha = \alpha(1) = m$.

4. <u>homomorphism</u>: Let $[\alpha], [\beta] \in \pi_1(s^1, (1,0))$. We want to show

$$f([\alpha] \cdot [\beta]) = f([\alpha]) + f([\beta])$$

.

Then $f([\alpha]) = n$ and $f([\beta]) = m$. So α is homotopic to the standard loop wrapping around the circle n times:

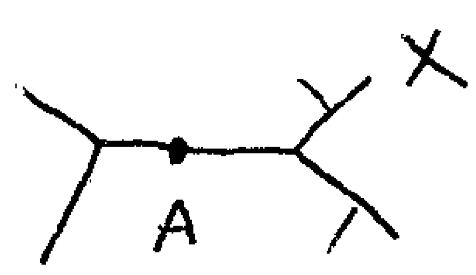
$$\alpha \simeq \alpha_n \text{ rel dI}$$
$$\beta \simeq \alpha_m \text{ rel dI}$$

$$\begin{aligned} f([\alpha] \cdot [\beta]) &= f([\alpha_n] \cdot [\alpha_m]) \\ &= f([\alpha \cdot \alpha_m]) \\ &= f([\alpha_(n+m)]) \\ &= n + m \\ &= f([\alpha_n]) + f([\alpha_m]) \\ &= f([\alpha]) + f([\beta]) \end{aligned}$$

Strong deformation retraction

If $A \subseteq X$, and $r : X \longrightarrow A$ is a retraction, then r is a *strong deformation retraction* if $I_x \simeq r$ rel A.

Example



There is an SDR from X to A.

Word

A *word* in the symbols a, b is a finite sequence of elements chosen from $\{a, a^{-1}, b, b^{-1}\}$.

Example

$aaba^{-1}bbbab^{-1}$ is a word.