LaTeX Topology Course Summary

Topology Overview and Goals

- Topological space- sets with some properties
- Ex: $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{T}^3$,...
- Topologists try to determine when 2 spaces are the " same "
- We say that the 2 spaces are homeomorphic when they are the " same "
- We set up invariants to determine if 2 spaces are homeomorphic
- These are functions whose input is a top space, and whose output is a number, group,...
- Can use the invariants to determine that 2 spaces are NOT homeomorphic; But can't use to prove they ARE homeomorphic

Chapter 0- Introduction and Background

Definition: An open interval in $\mathbb{R}$ is a set (a,b) = $\{x \in \mathbb{R} : a < x < b\}$

Definition: An open set U in $\mathbb{R}$ is a subset of $\mathbb{R}$ so that for every x in U, there is an open interval I so that $x \in I \subseteq U$

Definition: A map f from a topological space X to a topological space Y is continuous if and only if for every open set $U \subseteq Y$, the set $f^{-1}(U)$ is an open set in X.
*Note: Topologists ONLY care about continuous functions; Continuous functions are defined in terms of open sets.

Definition: A topology T on X is a collection of subsets of X so that

1. Each of {}, X are in T
2. The union of any collection of sets in T must also be in T
3. The intersection of any finite collection of sets in T must also be in T

Building Topologies from other Topologies

Definition: A basis B for a topology T on a space X so that every set in T can be obtained by taking arbitrary unions and finite intersections of elements of B.

Subspace Topology - Let X be a topological space with topology T. Let Y be a subset of X. Then T_Y=$\{Y \cap U : U \in T\}$
Product Topology - Let X and Y be topological spaces with topologies T_X and T_Y. The product topology on $X \times Y$ is the topology having basis, B=$\{U \times V : U \in T_X, V \in T_Y\}$

Quotient Topology - We write the quotient space as X^*. We produce the quotient space X by defining partitions of the space X. We partition space X into disjoint, nonempty subsets so that 2 points of X lie in the same subset iff we want to identify both of these points in the quotient space X^*.

*Note: These partitions defines an equivalence relation on X: x~y iff x and y are in the same set in the partition of X.

Define X^* to be the set of equivalence classes of X under ~, call it the quotient of X by ~. Thus, we write X^*=X/~

This defines a quotient map $p : X \to X^*$ sending $x \in X$ to the equivalence class containing $x \in X^*$

Basis: So $p : X \to X^*$ where p(x)= {equivalence class containing x} and the basis is $B_{X^*} = \{U \subseteq X^* : p^-1(U)$ is open in X}

Now we will look at the relations between topological spaces, the open sets within these spaces and the relations of groups the spaces build. In general, we do this by homeomorphisms (spaces), homotopys (functions), homomorphisms and isomorphisms (groups).

Definition: homeomorphic - Two spaces are homeomorphic if there exists a homeomorphism $f : X \to Y$

Definition: homeomorphism - A bijection $f : X \to Y$ such that f and f^-1 are both continuous.

Definition: bijection - A map $f : X \to Y$ that is both one-to-one (1-1) and onto.

Chapter 1- Fundamental Group

Definition: Two maps $f, g : X \to Y$ are homotopic if there exists a map $F : X \times I \to I$ such that for each $x \in X, F(x,0) = f(x)$ and $F(x,1) = g(x)$.

*Note: We say f is homotopic to g, written $f \simeq g$. And F is the homotopy between f and g.

Definition: A function $f : X \to Y$ is nulhomotopic if it is homotopic to a constant map.

Definition: A space X is path connected if for every $x_0, x_1 \in X$, there exists a path $f : I \to X$ so that $f(0) = x_0$ and $f(1) = x_1$

Definition: A map of pairs is $f : (X, A) \to (Y, B)$ which means $f : X \to Y$ and $f(A) \subseteq B$

By a map of pairs we know two maps can be homotopic relative to some open set:

Definition: Two maps $f, g : (X, A) \to (Y, B)$ are homotopic relative to A, written $f \simeq g$ rel. A, if there exists a homotopy $F : (X \times I, A \times I) \to (Y, B)$ so that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$, and for all $x \in A$, $F(x,t) = f(x)$ for every $t \in I$.

Definition: Two spaces X and Y are homotopy equivalent if there exists maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f = I_X$ and $f \circ g = I_Y$

Definition: A space which is homotopy equivalent to a point is called contractible.

Homotopy Classes of Loops and Fundamental Groups

Definition: Homotopy of Loops - Let α, β be loops in X with $\alpha(0) = \beta(0)$, then $\alpha \simeq \beta$ rel ∂I iff there's a map $f : I \times I \to X$ with $F(s, 0) = \alpha(s), F(s, 1) = \beta(s)$, F(0 or 1,t)=$\alpha(0)$.

Definition: The homotopy class of a loop α in X is based at $x_0 \in X$ is $[\alpha] = \{\beta : (I, \partial I) \to (X, x_0) : \alpha \simeq \beta \text{ rel } \partial I\}$

Definition: The constant loop is a space X based at $x_0 \in X$ is $c : (I, \partial I) \to (X, x_0)$ by $c(s) = x_0$ for all $s \in I$

Definition: Group - A group consists of a set G and an operation defined on that set $(G, \cdot)$ so that:

1. identity - There is an element $e \in G$ so that for every $g \in G$
$$e \cdot g = g \text{ and } g \cdot e = g$$
2. inverses - For all $g \in G$, there is $h \in G$ so that
$$g \cdot h = e \text{ and } h \cdot g = e$$
3. associativity - If $g, h, j \in G$, then
$$(g \cdot h) \cdot j = g \cdot (h \cdot j)$$

Ex's of groups: Trivial Group $\{1\}$, $\mathbb{R}$, $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{Z}$

Definition: The Fundamental Group of a space X based at x_0 is $\pi_1(X, x_0)$= set of all homotopy classes of all loops in X based at x_0. Write as $\{[\alpha] : \alpha : (I, \partial I) \to (X, x_0)\}$.

Furthermore:

Define $\pi_1(X, x_0)$ such that X is a top space and $x_0 \in X$

$\pi_1(X, x_0)$=$\{[\alpha] : \alpha : (I, \partial I) \to (X, x_0)\}$

$[\alpha] = \{\beta : (I, \partial I) \to (X, x_0) : \alpha \simeq \beta \text{ rel } \partial I\}$

$\alpha \simeq \beta$ rel $\partial I\}$ means there exists homotopy $H : (I \times I, \partial I \times I) \to (X, x_0)$ such that

H(s,0)=$\alpha(s)$

H(s,1)=$\beta(s)$

H(0,t)=x_0

H(1,t)=x_0

Then Define the operation on $\pi_1(X, x_0)$ by $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$

and
$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & \text{for} 0 \leq s \leq 1/2 \\ \beta(2s - 1) & \text{for } 1/2 \leq s \leq 1 \end{cases}$$

Isomorphism of Groups

Definition: Two groups $(G, *_G)$ and $(H, *_H)$ are homomorphic if there exists a function $f : G \to H$ so that for every $g_1, g_2 \in G$,
$$f(g_1 *_G g_2) = f(g_1) *_H f(g_2)$$

Definition: Two groups $(G, *_G)$ and $(H, *_H)$ are isomorphic if there exists a function $f : G \to H$ which is a bijective homomorphism.

*Note: We write this as $G \cong H$

Definition: A space X is called simply connected if it is path connected and $\pi_1(X, x_0) \cong \{1\}$ for every $x_0 \in X$

Up until this point, we have learned the basic terms and steps in order to understand the fundamental groups of various topological spaces. Now we will look at ways to conclude what the fundamental groups of these topological spaces are, and how this connects everything together. I will start with an example that shows how a space X with a loop homotopic to the constant loop is contractible. More specifically, it will show how the fundamental group of any contractible space X is isomorphic to the trivial group and therefore, simply connected. It will include various terms we have just reviewed.

Theorem

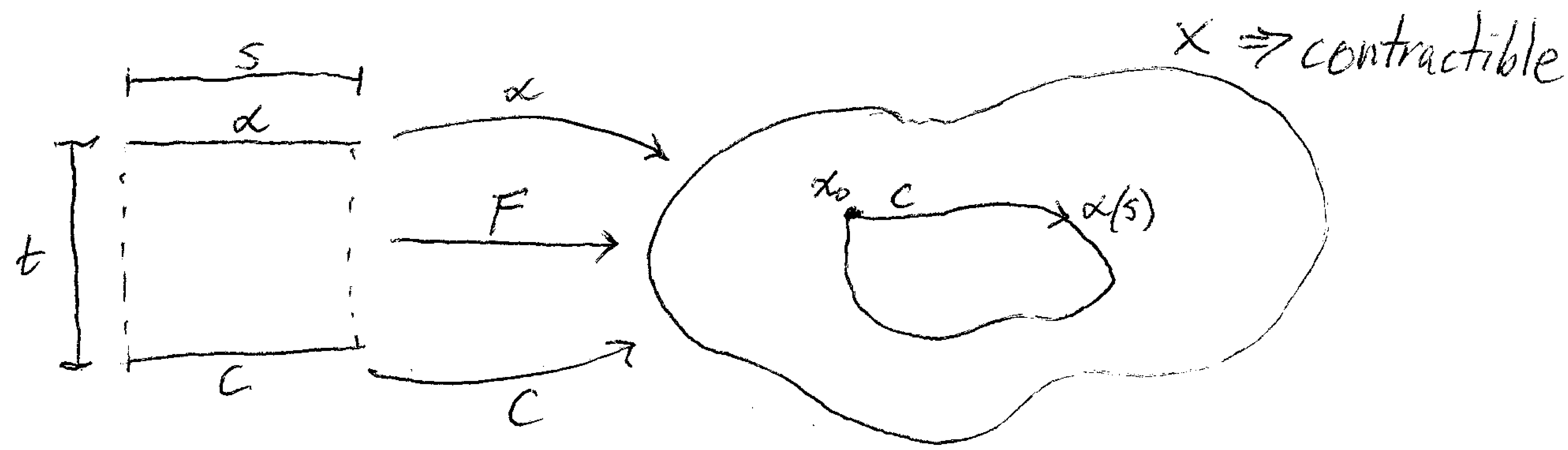
$\pi_1(X, x_0) \cong \{1\}$ if X is contractible.

Proof:

Assume that X is a contractible space and α is any loop based at x_0.

Assume α is homotopic to the constant loop $c : (I, \partial I) \to (X, x_0)$ by $c(s) = x_0$ for all $s \in I$. So $\alpha \simeq c$ rel ∂I.

We want to show $\pi_1(X, x_0) \cong \{1\}$.



* From HW 4.#1 Given: Homotopy $F:(I\times I, \partial I\times I) \to (X, x_0)$

Since $\alpha \simeq c$ rel ∂I, we know that any loop α based at x_0 in X is homotopic to the constant map.

$\pi_1(X, x_0)$ = set of all homotopy classes of all loop in X based at x_0

$= \{[\alpha] : \alpha : (I, \partial I) \to (X, x_0)\}$

Since we know X is contractible there is only one set of homotopy class of loops based at x_0. This will be called $[\alpha]$.

And since $\pi_1(X, x_0)$ is the set of all homotopy classes of all loops in X based at x_0, then $\pi_1(X, x_0) = \{[\alpha]\}$

Now we can show $\pi_1(X, x_0) \cong \{1\}$ by $\{[\alpha]\} \cong \{1\}$.

To do this, we need a function $f : \{[\alpha]\} \to \{1\}$ which is a bijective homomorphism.

So we have two groups $(\{[\alpha]\}, *_\alpha)$ and $(\{1\}, *_1)$.

Define $f : \{[\alpha]\} \to \{1\}$ by f(e)=1 such that $e \in \{[\alpha]\}$.

Now check that f is a bijection:

Onto: Every element of the range $\{1\}$ must have a pre-image in the domain, which 1 clearly does.

1-1: Every element of the domain $\{[\alpha]\}$ has exactly 1 element in the range $\{1\}$, which it clearly does since both are 1 element sets.

Therefore, f is a bijection.

Now we must show f is a homomorphism.

We defined $f : \{[\alpha]\} \to \{ 1\}$ by f(e)=1.

Since the domain has one element $[\alpha]$ we can write the two groups, $(\{[\alpha]\}, *_\alpha)$, $(\{1\}, *_1)$ as $f([\alpha] *_\alpha [\alpha]) = f([\alpha]) *_1 f([\alpha])$

So $1 = 1 *_1 1$ since we defined f by f(e)=1.

And since $1 *_1 1=1$ and $1 *_1 1$ must be a group then the operation $*_1$ must be multiplication.

So $1 \cdot 1 = 1$, and $\{[\alpha]\}$ is homomorphic to $\{1\}$, $\{[\alpha]\} \cong \{1\}$

Therefore, since f is a bijective homomorphism and $\pi_1(X, x_0) = \{[\alpha]\}$, then

$$\pi_1(X, x_0) \cong \{1\}$$

□

Another important theorem we have learned is the Unique Path Lifting Lemma. This is especially important to understand when we get to covering spaces and projection. It gives us another invariant and basically allows us to prove that the fundamental groups of some complex spaces, or even simple spaces are isomorphic to various groups. In class we proved that $\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$, which I will review and write a sketch proof of.

Unique Path Lifting Lemma: Given $\alpha : (I, \partial I) \to (\mathbb{S}^1, (1, 0))$, there is a unique path $\widetilde{\alpha} : I \to \mathbb{R}$ with $\widetilde{\alpha}(0) = 0$ and $w \circ \widetilde{\alpha}(s) = \alpha(s)$ for all $s \in I$

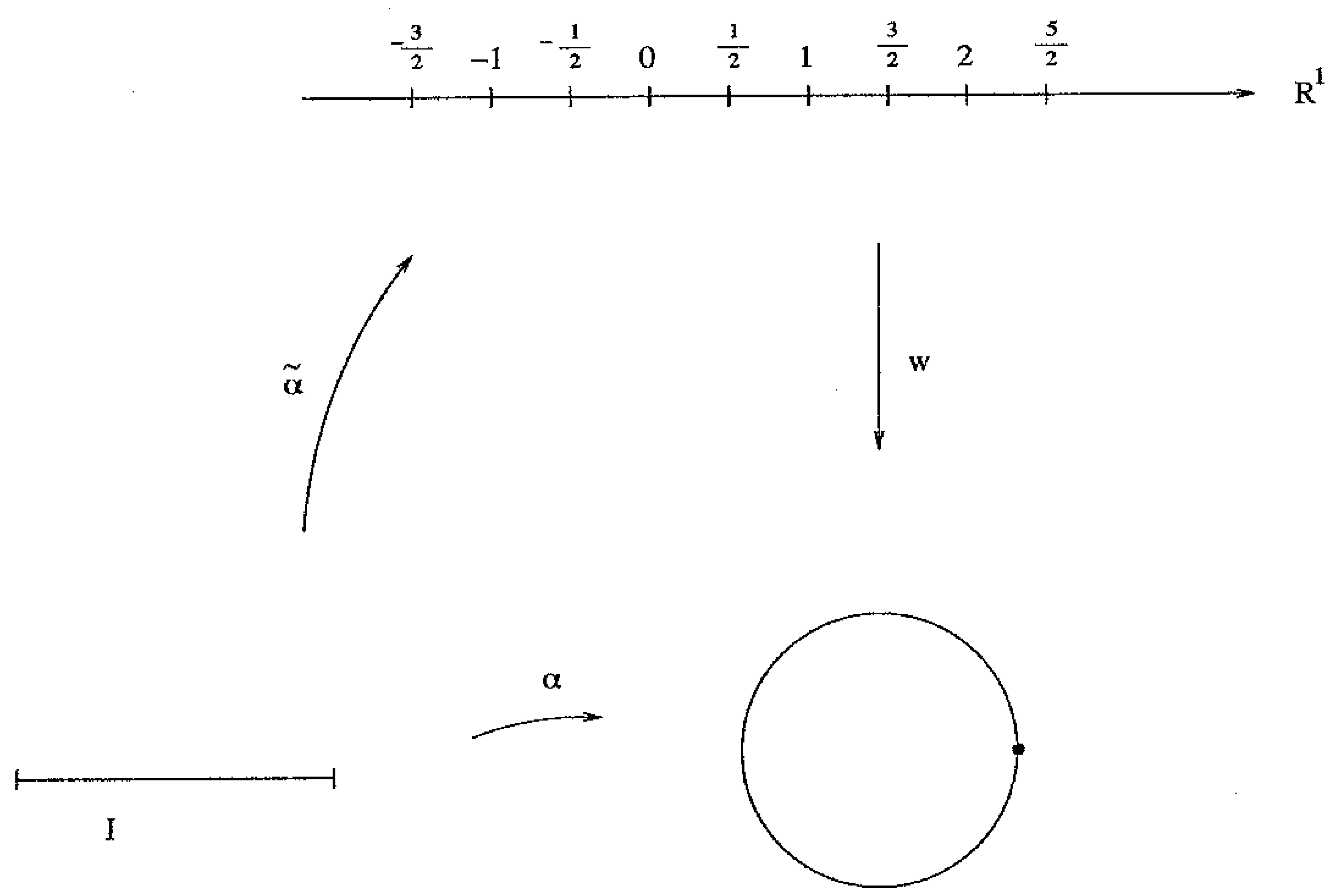


Figure 1: Lifting a loop from the circle to the real line.

Theorem

$\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$

Proof: This will be a sketch proof using the Unique Path Lifting Lemma.

First off, our final goal is to define a map $f : \pi_1(\mathbb{S}^1, x_0) \to \mathbb{Z}$ that is a bijective homomorphism.

Let $\alpha : (I, \partial I) \to (\mathbb{S}^1, (1,0))$ be a loop in $\mathbb{S}^1$. In order to define f, we need to associate an integer to α.

So let $f(\alpha)$= number of times image of α wraps around $\mathbb{S}^1$ in clockwise direction.

This is where the Unique Path Lifting Lemma comes in.

Now we can look at the diagram above to see that if w was invertible, we could define $\widetilde{\alpha} = w^-1 \circ \alpha$. However, w is not invertible.

But w is invertible in sub-intervals of $\mathbb{R}$ of length less than 1.

So we make w a local homeomorphism.

Thus, we can inductively suppose that $\widetilde{\alpha}(s)$ is defined in sub-intervals less than 1.

Then we use the Unique Path Lifting Lemma to lift α to $\widetilde{\alpha} : I \to \mathbb{R}$ and define the function $f(\alpha) = \widetilde{\alpha}(1) \in \mathbb{R}$

Finally we show

1. $\widetilde{\alpha}(1) \in \mathbb{Z}$ by letting a homotopy class $[\alpha] \in \pi_1(\mathbb{S}^1, (1,0))$ and defining a function for w so that $f([\alpha])$=an integer.

2. Then check if f is well-defined: If $[\alpha] = [\beta]$ then $f(\alpha) = f(\beta)$ by using a homotopy onto $\mathbb{S}^1$ and defining a function from loops in $\mathbb{S}^1$ to the integers to show that the integer of α=integer of β.

3. Then check if f is 1-1: If $f([\alpha]) = f([\beta])$ then $[\alpha] = [\beta]$ by using the lemma again to define a homotopy for $\widetilde{F}$ then showing that our results are equivalent to the homotopy of F.

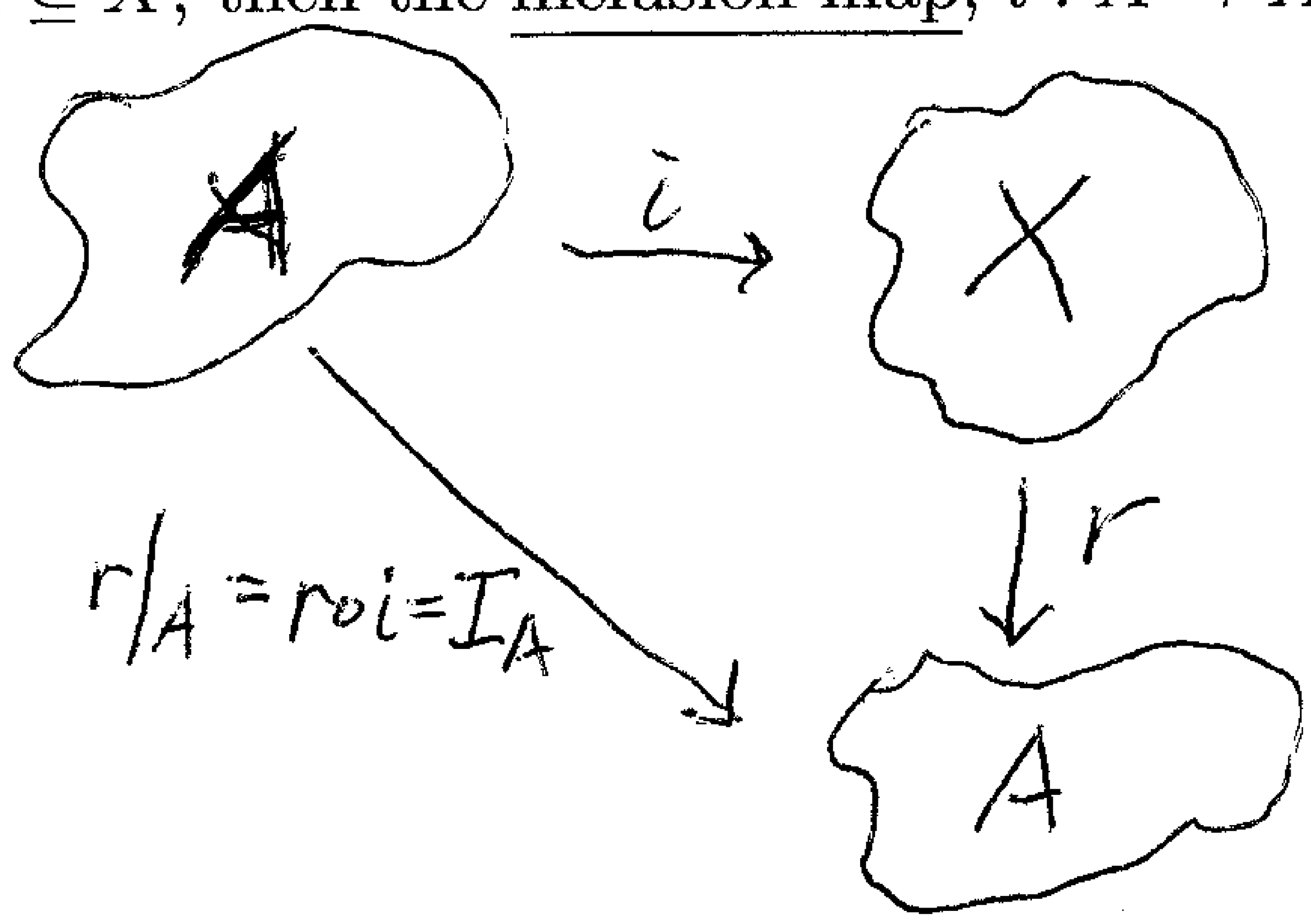
4. Then check if f is onto by showing that the function of the homotopy class $[\alpha]$ will give us an integer.

5. Finally, check to see if f is homomorphism by showing there is a function so the elements $[\alpha], [\beta] \in \pi_1(\mathbb{S}^1, (1,0))$ are equivalent by each group's operation.

□

Definition: If $A \subseteq X$, then a <u>retraction</u> of X onto A is a continuous function $r : X \to A$ with $r|_A$=I_A (alternately $r \circ i = I_A$)

Definition: If $A \subseteq X$, then the <u>inclusion map</u>, $i : A \to X$ is i(a)=a for all $a \in A$.



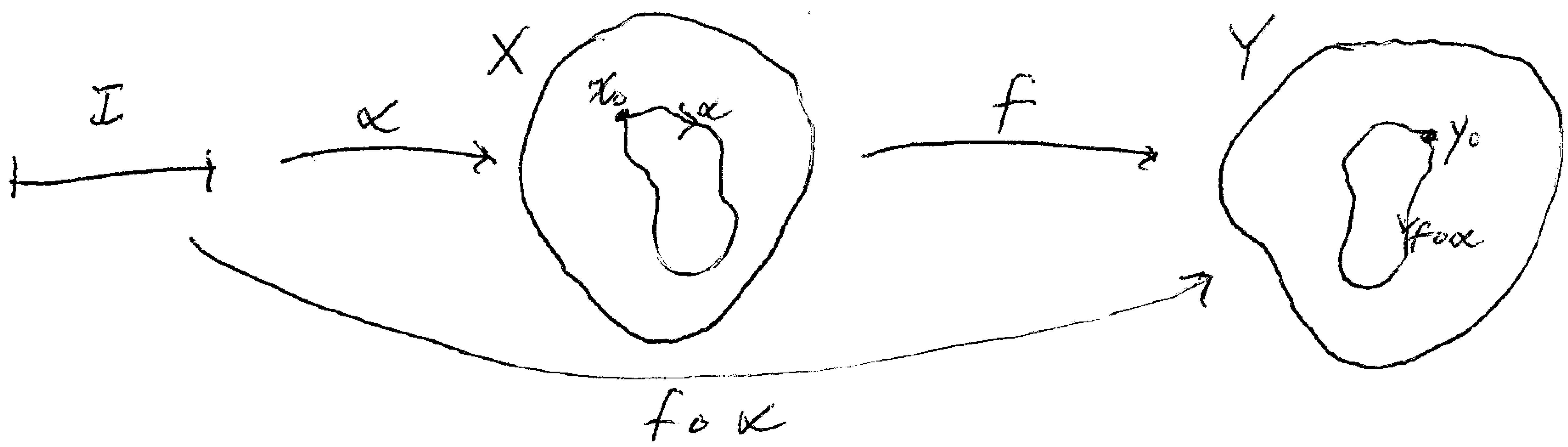
Here is an important definition/idea that helps us to calculate fundamental groups.

Definition: If $A \subseteq X$ and $r : X \to A$ is a retraction, then r is a <u>strong deformation retraction (sdr)</u> such that $I_X \simeq r$ rel A.

Definition: Induced Homomorphism on π_1 - Suppose $f : (X, x_0) \to (Y, y_0)$, and suppose $\alpha : (I, \partial I) \to (X, x_0)$ is a loop in X.

Then $f \circ \alpha : (I, \partial I) \to (Y, y_0)$ is a loop in Y.

This gives a map $f_*([\alpha]) = [f \circ \alpha]$.

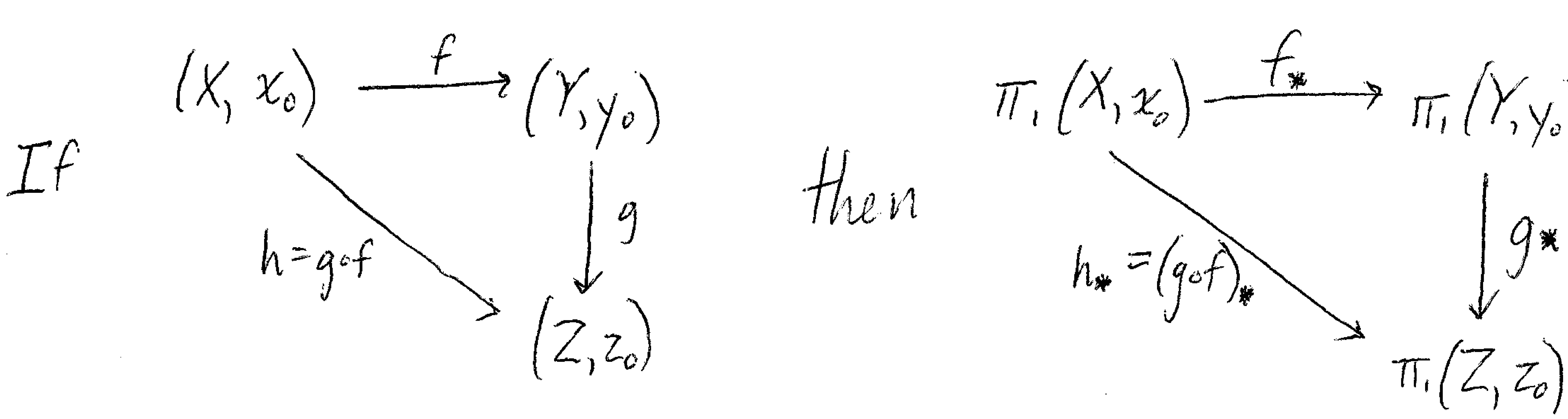


We are given a couple properties for the induced homomorphism on π_1:

Prop. 1: f_* is a homomorphism.

Prop. 3.24.A:

1. If $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ then
$$(g \circ f)_* = g_* \circ f_*$$



2. If $I_X : (X, x_0) \to (X, x_0)$ is the identity map on spaces, then $(I_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity map on groups.

Prop. 3.24.B: If $f \simeq g : (X, x_0) \to (Y, y_0)$ rel x_0, then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

Theorem

If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

The following Corollary logically stems from the theorem and properties above. If a space X is homeomorphic to a space Y then there exists a retraction from one space to the other. Specifically, there is a strong deformation retraction because the homeomorphism implies that the retraction would be continuous. Therefore, their fundamental groups are isomorphic.

An important Corollary to take away from induced homomorphism: Theorem

If $r : X \to A$ is a strong deformation retraction ($I_X \simeq r$) then $r_* : \pi_1(X, x_0) \cong \pi_1(A, x_0)$

Proof:

Suppose $r : X \to A$ is a sdr.
Let $i : A \to X$ be inclusion map.
Since r is a retraction,

$$r \circ i = I_A$$
$$(r \circ i)_* = r_* \circ i_* = (I_A)_* \qquad \text{(Prop 3.24.A.1)}$$

Since r is a sdr,

$$r \simeq I_X$$
$$r_* = (I_X)_* \qquad \text{(Prop 3.24.B)}$$

$$i \circ r \simeq i \circ I_X \qquad \text{(Composition of functions)}$$

$$i_* \circ r_* = i_* \circ (I_X)_* \qquad \text{(Prop 3.24.A.1)}$$

So $r_* : \pi_1(X, x_0) \cong \pi_1(A, x_0)$ is invertible (Prop 3.24.A.2)

And r_* is a homomorphism because it's the induced map (Prop 1)

□

More Group Theory

Prop: Given 2 groups $(G, \cdot_G), (H, \cdot_H)$, you can define a new group $G \times H$ as follows:
The elements of $G \times H$ are ordered pairs (g,h) so that $g \in G$ and $h \in H$.
To multiply the elements of $G \times H$: $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H h_2)$.

The above property allows us to break down more complex topological spaces in order to more easily find their fundamental groups. This points to the following theorem.

Theorem

Let $x_0 \in X, y_0 \in Y$. Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Ex: Torus - $\mathbb{T}^2$

$\mathbb{T}^2$=$\mathbb{S}^1 \times \mathbb{S}^1$

$\pi_1(\mathbb{T}^2, (x_0, y_0)) \cong \pi_1(\mathbb{S}^1 \times \mathbb{S}^1, (x_0, y_0))$

$\cong \pi_1(\mathbb{S}^1, x_0) \times \pi_1(\mathbb{S}^1, y_0)$

$\cong \mathbb{Z} \times \mathbb{Z}$

Group Presentation

A free group of rank 2, written F_2, has 2 generators a and b, that define elements/reduced words/relations of the group. We can multiply words by concatenation then cancellation to find reduced words.

$F_2 \cong \langle a, b| \qquad \rangle$

Defintion: Finite Presentation for a group G is $\langle g_1, g_2, ..., g_n | R_1, R_2, ..., R_m \rangle$

Definition: A group is abelian if ab=ba for all $a, b \in G$

Ex: Give a finite presentation for $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, the fundamental group of the 3-torus.

We want to find the relations of a group with 3 generators: a, b, c.

There should be:

ab=ba by ab$\overline{a}\overline{b}$

bc=cb by bc$\overline{b}\overline{c}$

ac=ca by ac$\overline{a}\overline{c}$

We know a$\overline{a}$, $b\overline{b}$, $c\overline{c}$ are the identity elements

Lets try the relation abc$\overline{a}\overline{b}\overline{c}$; the relation with all 3 generators.

We know bac$\overline{a}\overline{b}\overline{c}$ by ab$\overline{a}\overline{b}$

Then, bca$\overline{a}\overline{b}\overline{c}$ by ac$\overline{a}\overline{c}$

a and $\overline{a}$ cancel and we're left with bc$\overline{b}\overline{c}$

So abc$\overline{a}\overline{b}\overline{c}$ can't be a relation.

Therefore, $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \cong \langle a, b, c | ab\overline{a}\overline{b}, bc\overline{b}\overline{c}, ac\overline{a}\overline{c} \rangle$

Ways to Calculate $\pi_1(X, x_0)$

Theorem

π_1(bouquet or wedge of circles with 1 point on each identified)$\cong \mathbb{Z} * \mathbb{Z} * ... * \mathbb{Z} = \mathbb{Z}(n) \cong F_n$

Ex's of calculating π_1 of some spaces:

Recall SDR Corollary

1. $\mathbb{T}^2 - pt$

   $\mathbb{T}^2 - pt \xrightarrow{sdr}$ (wedge of 2 circles)

   $\pi_1(\mathbb{T}^2 - pt, x_0) \cong \pi_1$(wedge of 2 circles)

   $\cong \mathbb{Z} * \mathbb{Z}$

2. $\mathbb{R}^2 - a, b$ where a and b are 2 distinct points

   $\mathbb{R}^2 - a, b \xrightarrow{sdr}$ (wedge of 2 circles)

   $\pi_1(\mathbb{R}^2 - a, b, *) \cong \pi_1$(wedge of 2 circles)

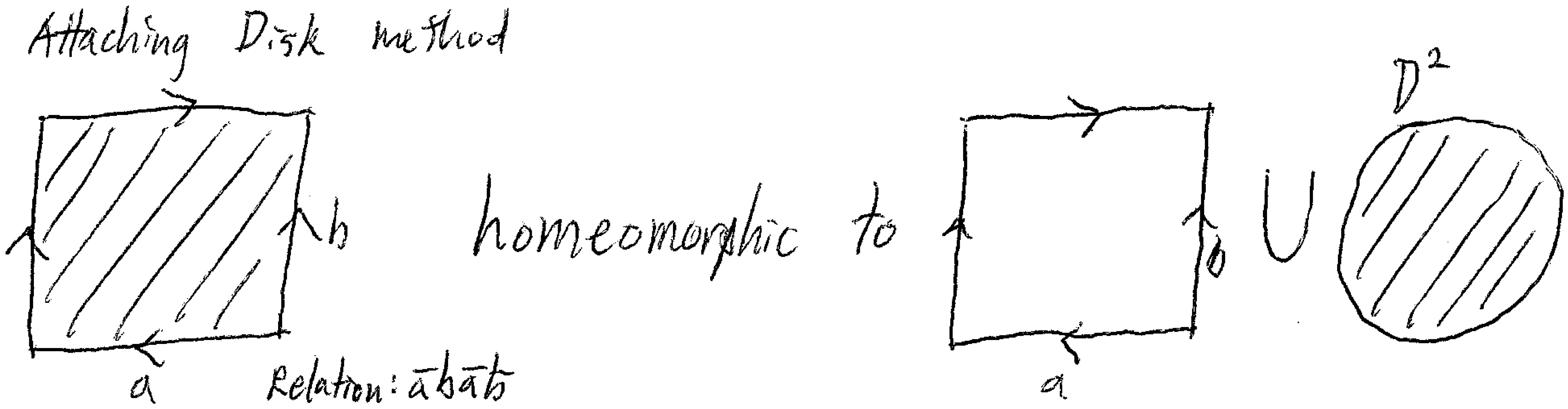
   $\cong \mathbb{Z} * \mathbb{Z}$

3. $\mathbb{R}^3 - a, b$ where a and b are 2 distinct points

   $\mathbb{R}^3 - a, b \xrightarrow{sdr}$ (wedge of 2 spheres)

   And since we know $\pi_1(\mathbb{S}^2, *) \cong \{1\}$, then $\pi_1(\mathbb{R}^3 - a, b, *) \cong \{1\}$

4. X=Klein Bottle



   $\pi_1(X, *) = \langle a, b | \overline{a} b \overline{a} \overline{b} \rangle$

One way we can calculate the fundamental groups of spaces is by sdr's, as we have done above. Essentially, sdr's take continuous functions within a given space and map the points within that space to simpler spaces so we can define a fundamental group. For example, we take $\mathbb{R}^3$-1 point and can conclude it can be retracted to a sphere, whose fundamental group is $\{1\}$, the trivial group. We can also use the attaching disk method to solve fundamental groups. Attaching a disk using a function kills the loop in a space X, by essentially molding the disk and using its boundary to fit the space. This is what we did for the klein bottle above. In retrospect, sdr'ing is conceptually the most efficient and probably best way to finding fundamental groups of certain spaces. So, in general, if you are trying to find the fundamental group of complex spaces then see if there is a sdr and possibly break it down into components of the larger space and combine them to solve the fundamental group.

Chapter 2 - Covering Spaces

Definition: $p : E \to B$ is a <u>covering space projection</u> if E and B are path connected and for all $b \in B$ there exists a neighborhood U of b such that each component of $p^-1(U)$ maps homoemorphically onto U via p.

Theorem (Theorem 5.3.1)

If X is simply connected and locally path connected, and $p : E \to B$ is a covering $f : X \to B$ is continuous, then there exists a continuous lift $\widetilde{f} : X \to E$ so that $p \circ \widetilde{f} = f$

Definition: If $p : E \to B$ is a covering space projection and E is simply connected then E is called the <u>universal cover</u> of B.

Notation: $p : \widetilde{X} \to X$, where $\widetilde{X}$ is universal cover of X.