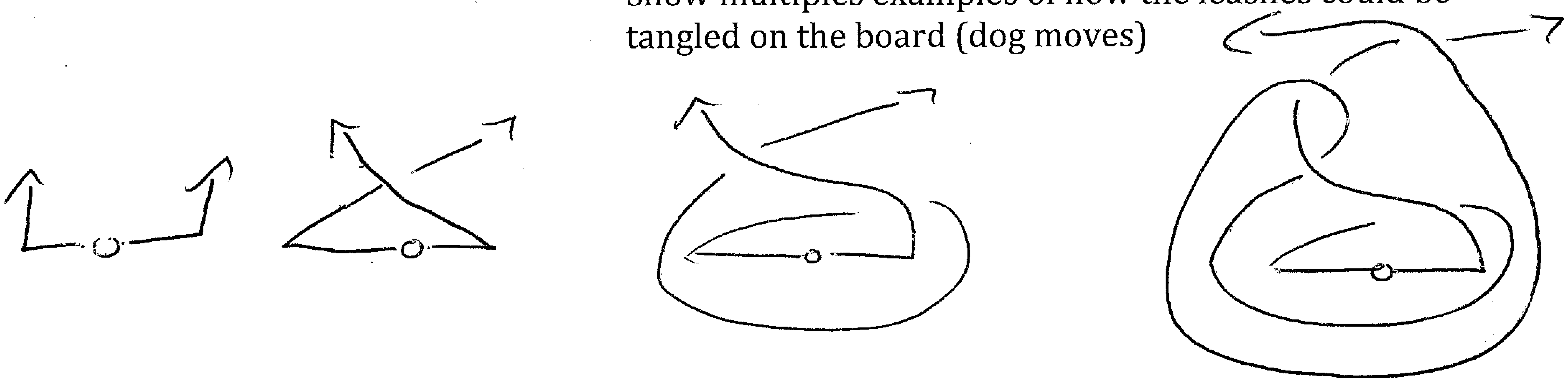
Taking the Wursts for a Turn:

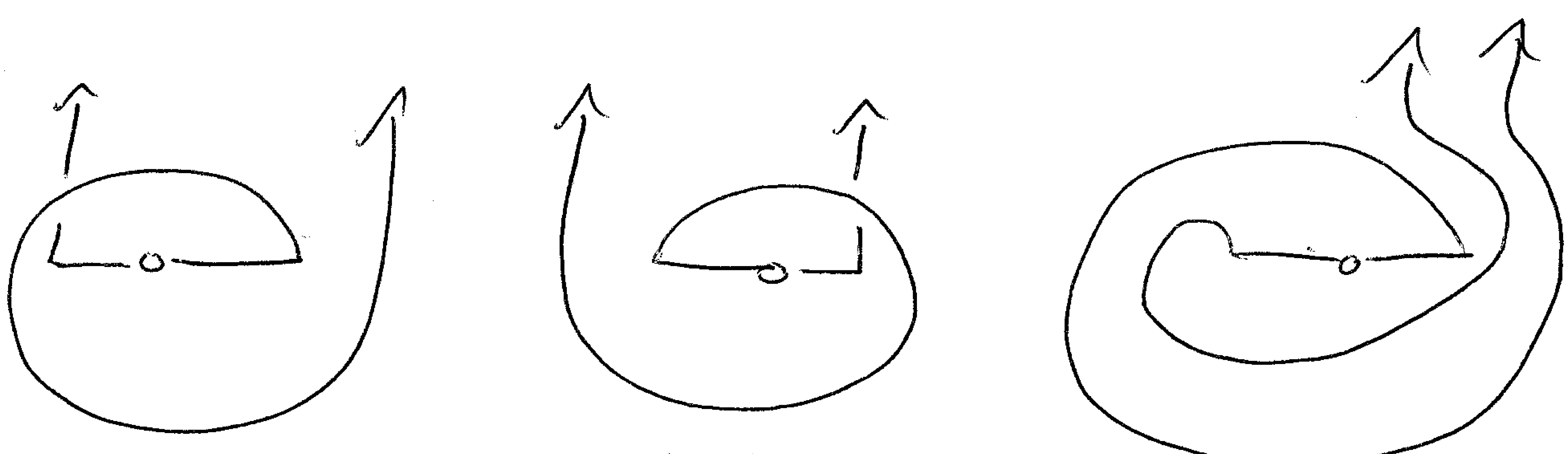
Algebra, Topology and Dachshunds

Richard P Kubelka

- Intro
  - He was out walking his two dachshunds Sascha and Fritz one day and he began to become entangled in their leashes as they moved around.
  - He kept having to switch the leashes from hand to hand, step over the leashes, or completely untangle the leashes constantly.
  - He was fed up with having to do this so he began to think of his problematic situation in terms of topology to determine whether he could untangle himself from the leashes without stepping over them or changing hands→ moves that he says are "cheating".
  - Eventually he found that there were certain moves he could do which would untangle the leashes without him having to cheat. The led him on a topological mind train of which is the source of the paper.
  - DEMONSTRATION OF WALKING TWO DOGS
  - Now, he determined that of all the possible things the dogs could do, it seemed to appear that there were only a certain "number" of 'dog moves' and a certain number of 'walker moves'
    - Dog Moves
      - Show multiples examples of how the leashes could be tangled on the board (dog moves)



    - Walker Moves
      - Show multiple examples of how the walker can unravel the leashes via the walker subgroup on the board(the subgroup of D2 that he can resolve by his actions)

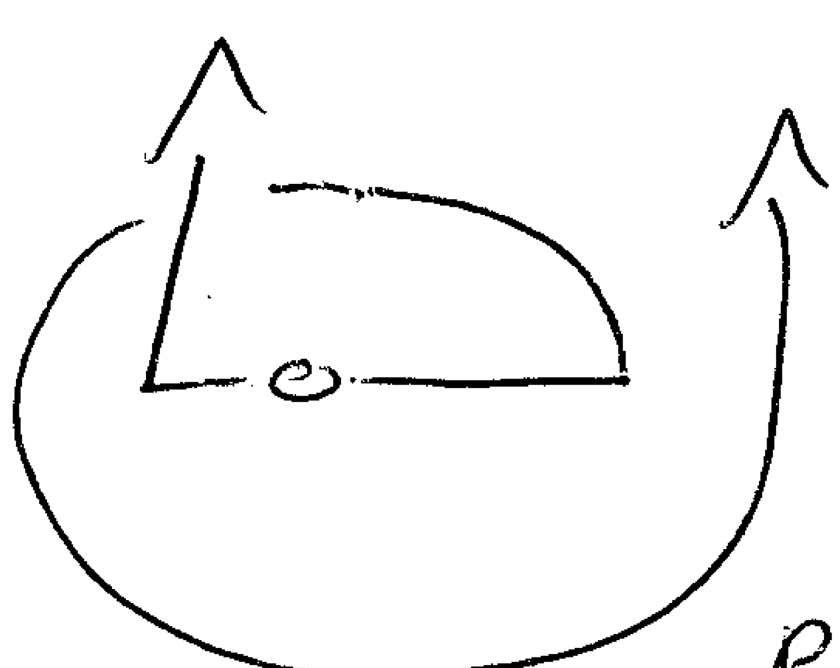


R.H L.H

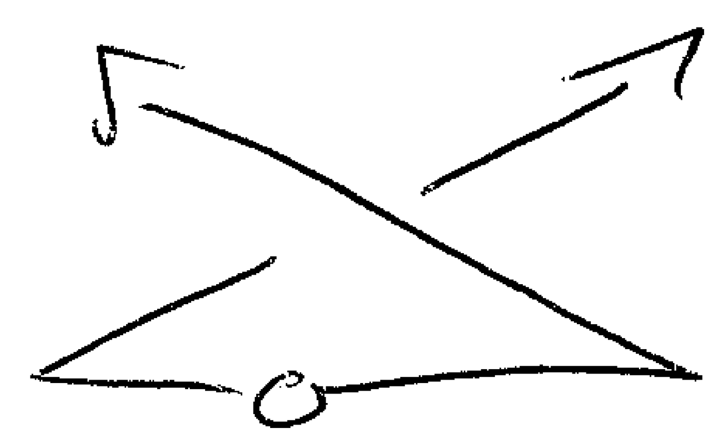
- Said that this problem of trying to 'unravel' the leashes reminded him of Artins Braid Groups.
  - The nice thing about artins braid groups was that this would allow him to solve the problem of whether or not he had thought up all possible dog moves.
  - This will be explained later on, but the basic idea is that this problem of tangling leashes can be thought of as braids with three string→ where one is Dog 1 one is the walker (in the middle) and one is Dog 2.
- Briefly explain this → approach
  - Ordinary braids can be factored into a product of simpler braids. This is what he wanted to do with the dog leashes. He wanted to see if he could 'factor' the tangled leashes into a product of simpler dog moves.

- What he hopes to achieve
  - To show that he could resolve almost every tangle the dogs could weave by a set of walker moves (a subgroup of D2)
    - Show D_2 has 'a presentation with two generators and one relation

- The 2-Dog Group D^2
  - Includes all possible entanglements of the dogs and their leashes around themselves and him
    - Describe and show the 2 basic moves R and T and mirrors

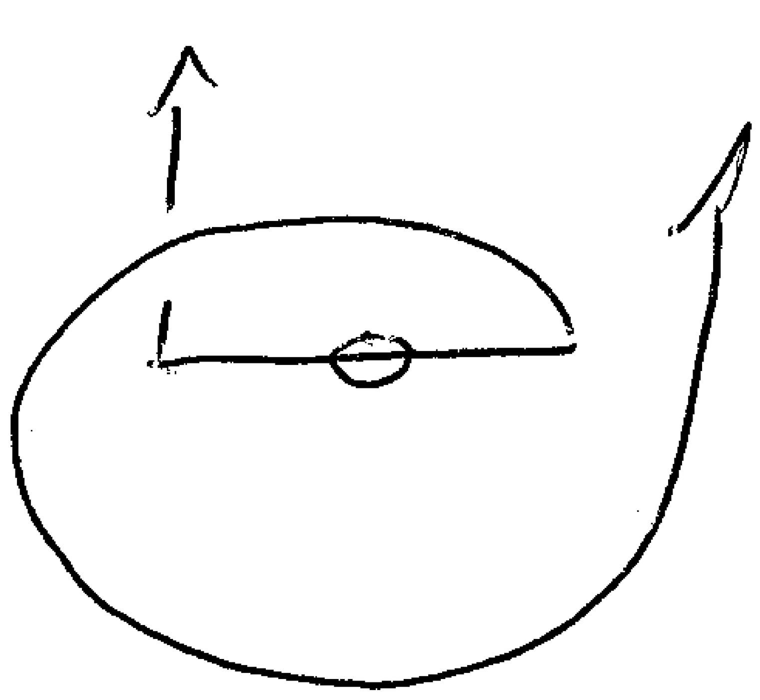


R

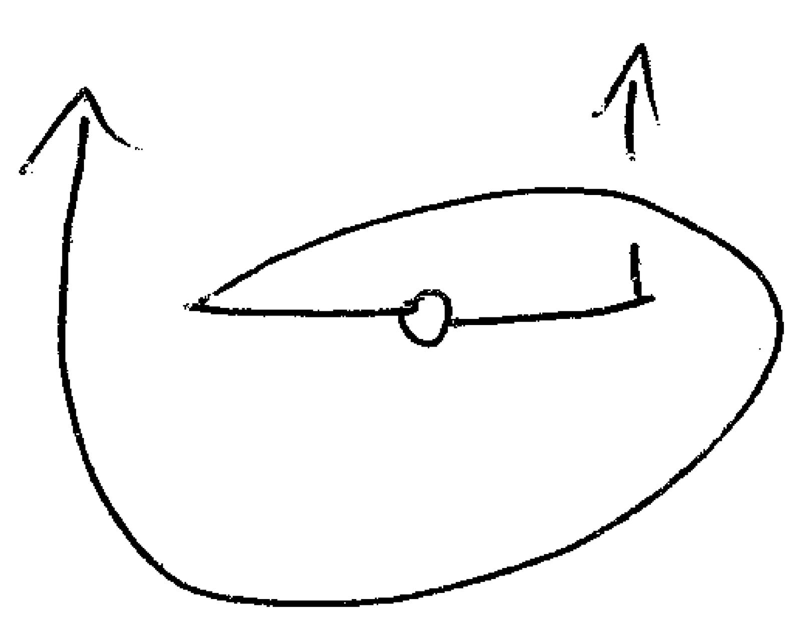


T

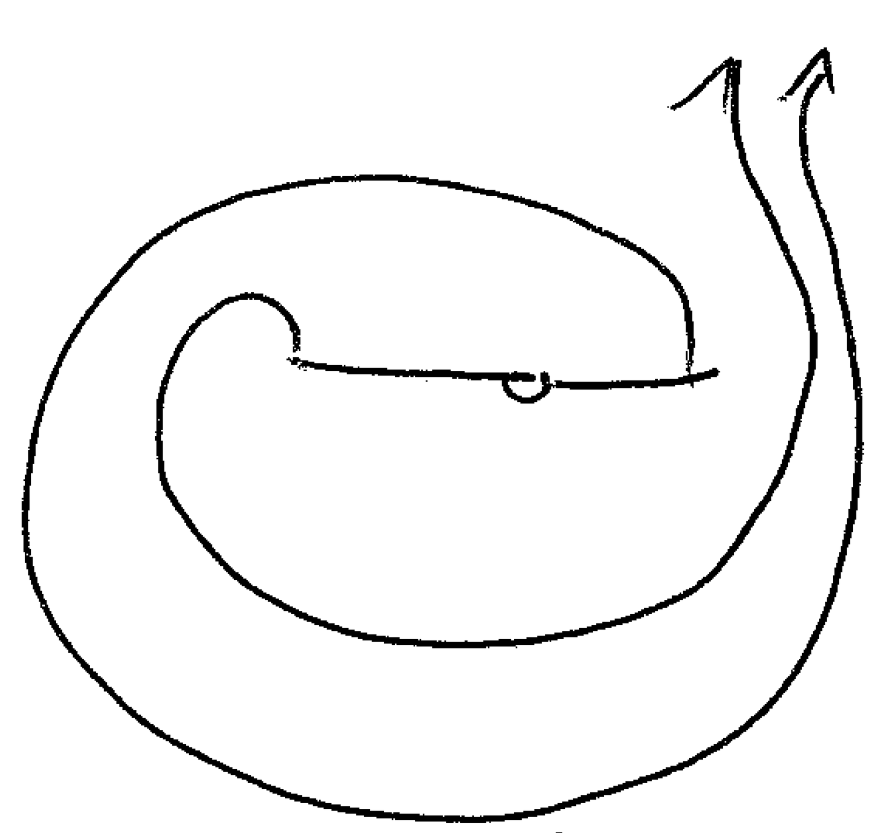
- The Walker Subgroup W (subgroup of D_2) Subset of tangles that → he could resolve himself
  - Consists of all the moves that the walker can make
    - ~~The left hand does not know what the~~ right hand ~~is doing~~
  - Describe and show the two basic moves rho, lambda and y



ρ



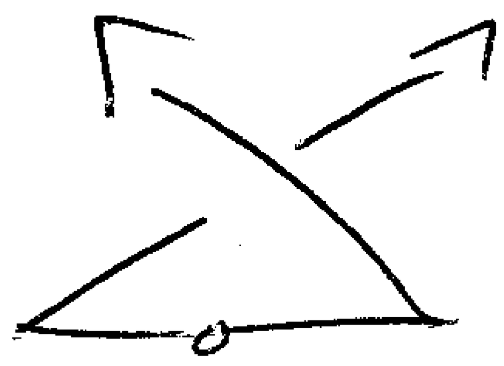
λ



γ

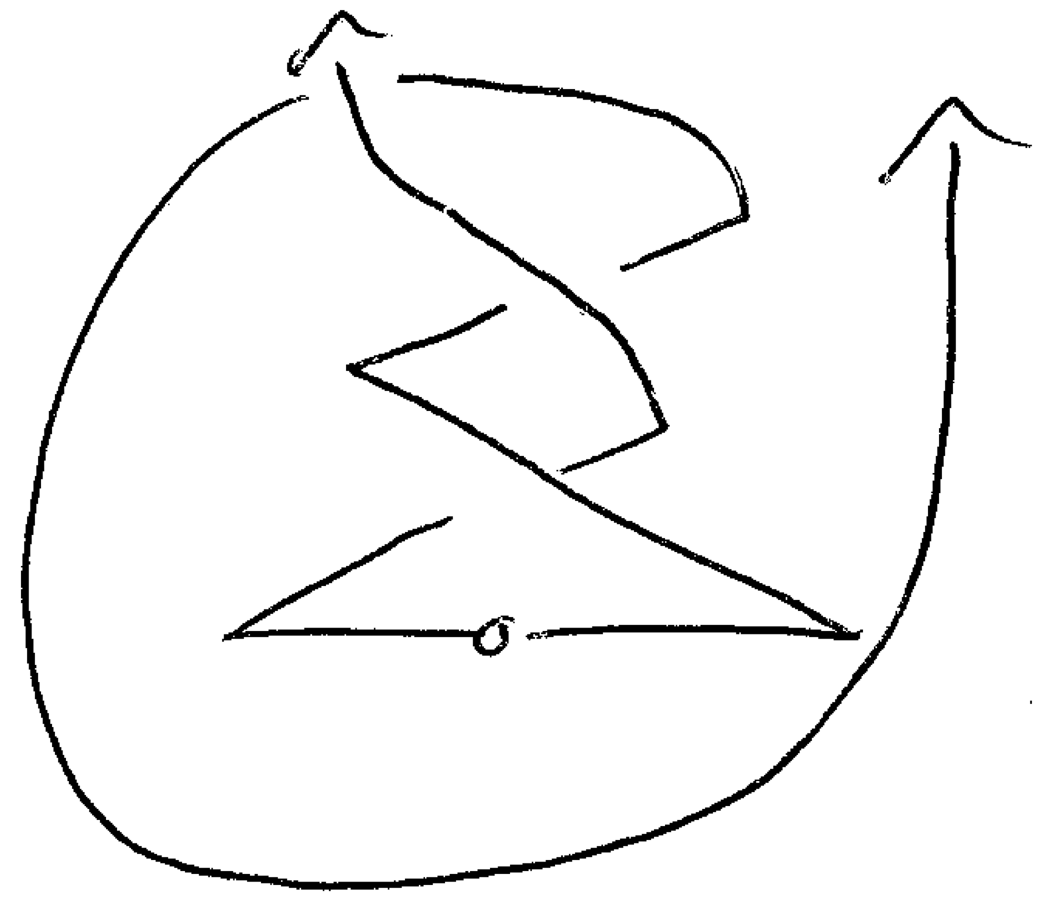
γ^{-1}

    - He needs an operator on the group
        - Describes this as concatenation (Figure 3)
            - Show different moves such as T^2R



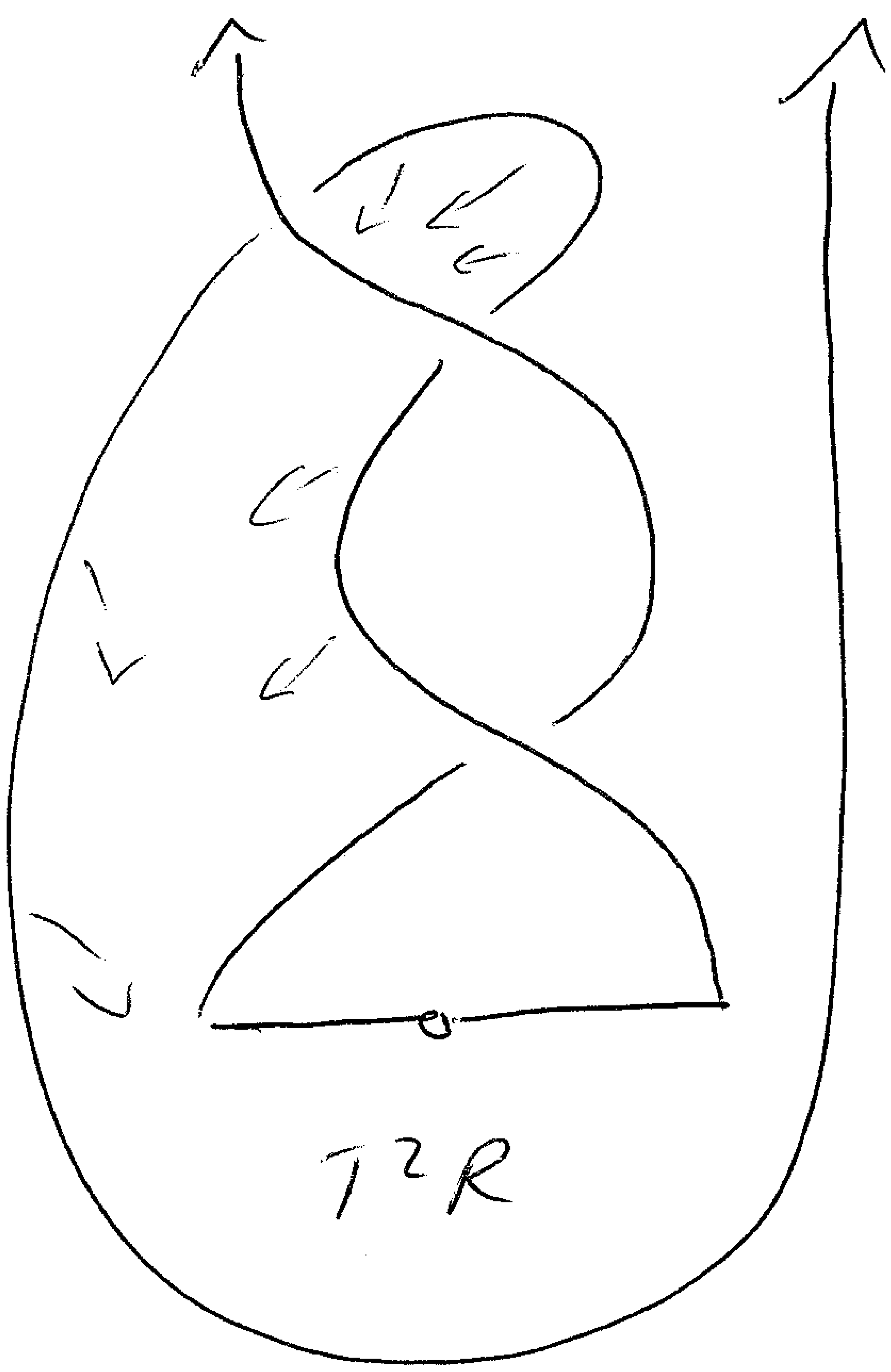
T

T^2



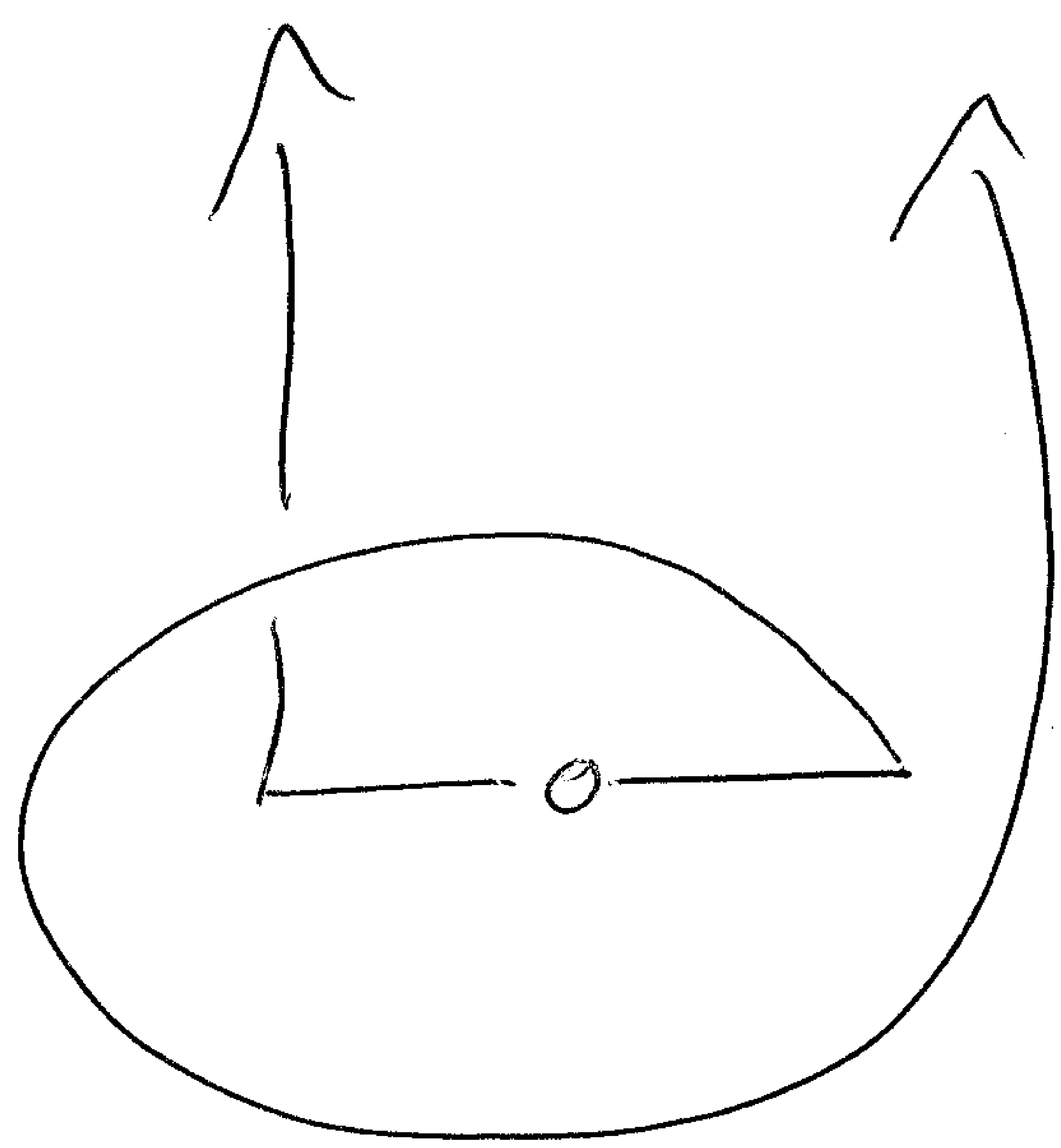
T^2R

- Relation on D^2
    - $p=T^2R$
        - This is the only relation.
        - Demonstrate



T^2R

$\Rightarrow$



ρ

~~X~~ shows that ρ is included in D^2

- Introduce Main Theorem pg 3 write on board

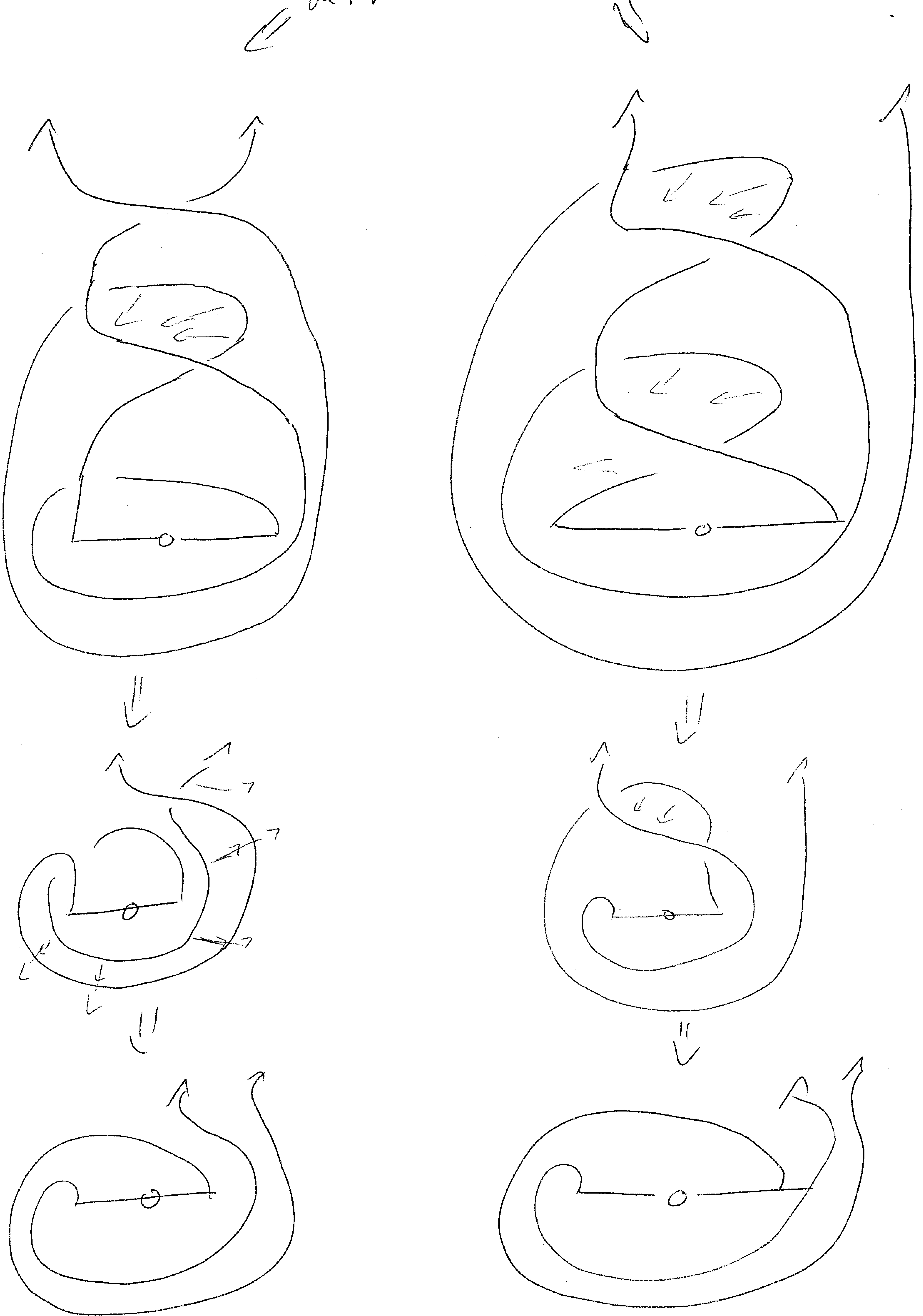
(i) $D_2 = \langle R, T \mid (RT^2)(TR)^{-2} \rangle$ relation is $(RT)^2 = (TR)^2$

on next page

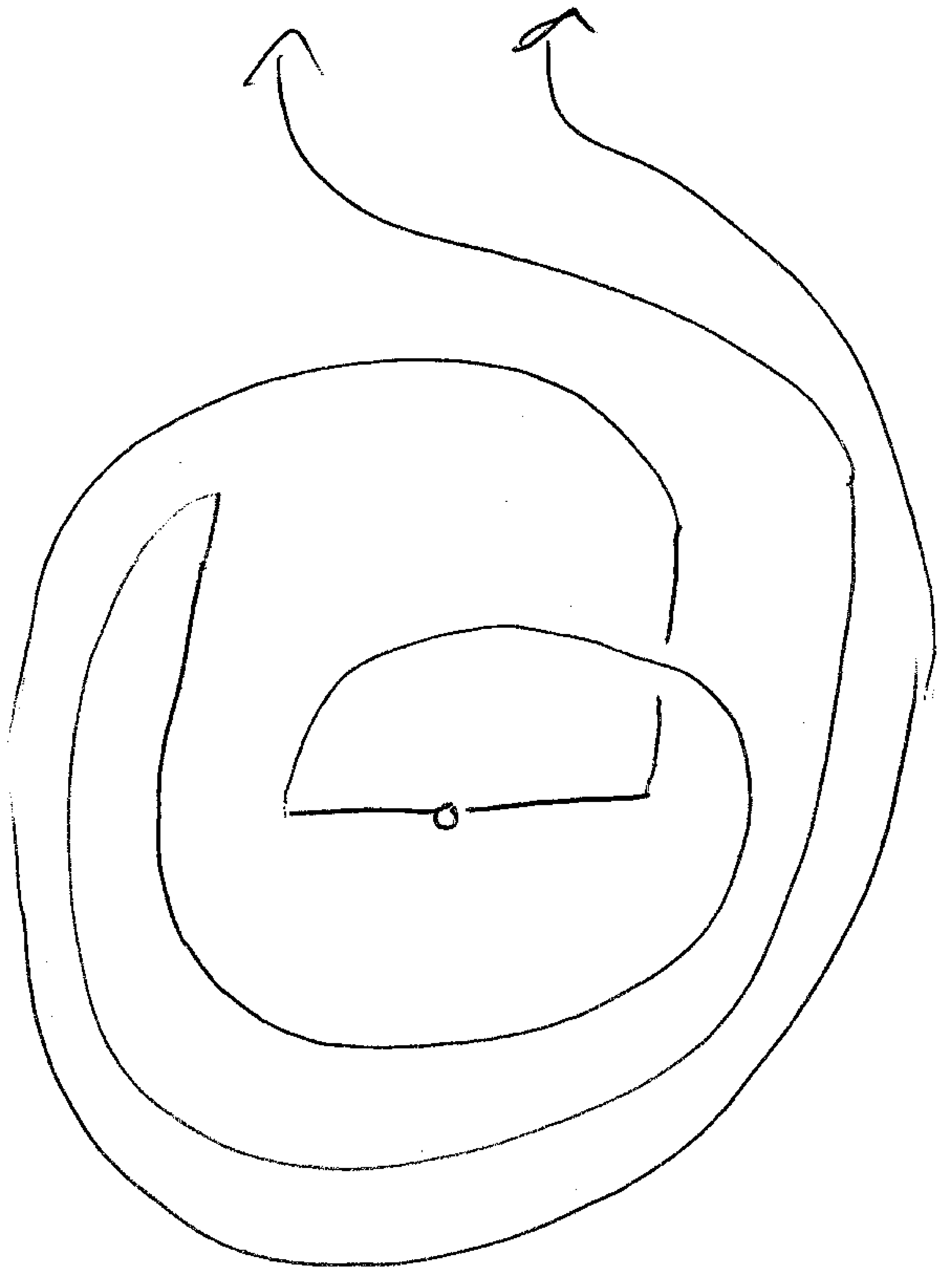
(ii) $W = \langle \lambda, \rho, y \mid \gamma\lambda\gamma^{-1}\lambda^{-1}, \gamma\rho\gamma^{-1}\rho^{-1} \rangle$
relations are $\lambda\gamma = \gamma\lambda$ and $\rho\gamma = \gamma\rho$

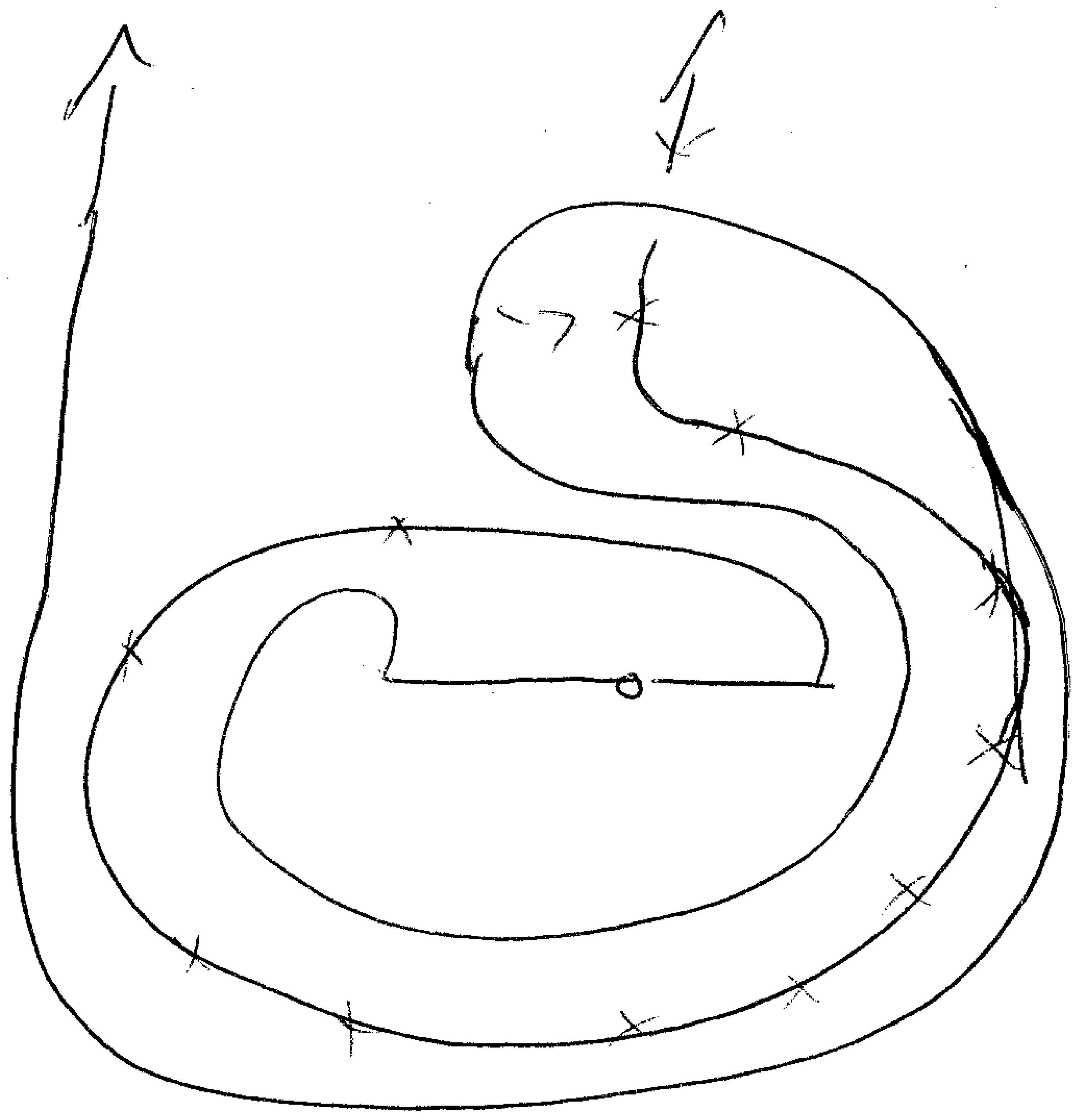
(iii) $F_2 = \langle \lambda, \rho \rangle \leq W$ is the free group on generators λ and ρ.
↳ means the right hand doesn't know what the left hand is doing.
$\Rightarrow$ no relations $\Rightarrow$ no amount of doing λ or ρ or any combination will get you back where you started.

$RTRT = TRTR$

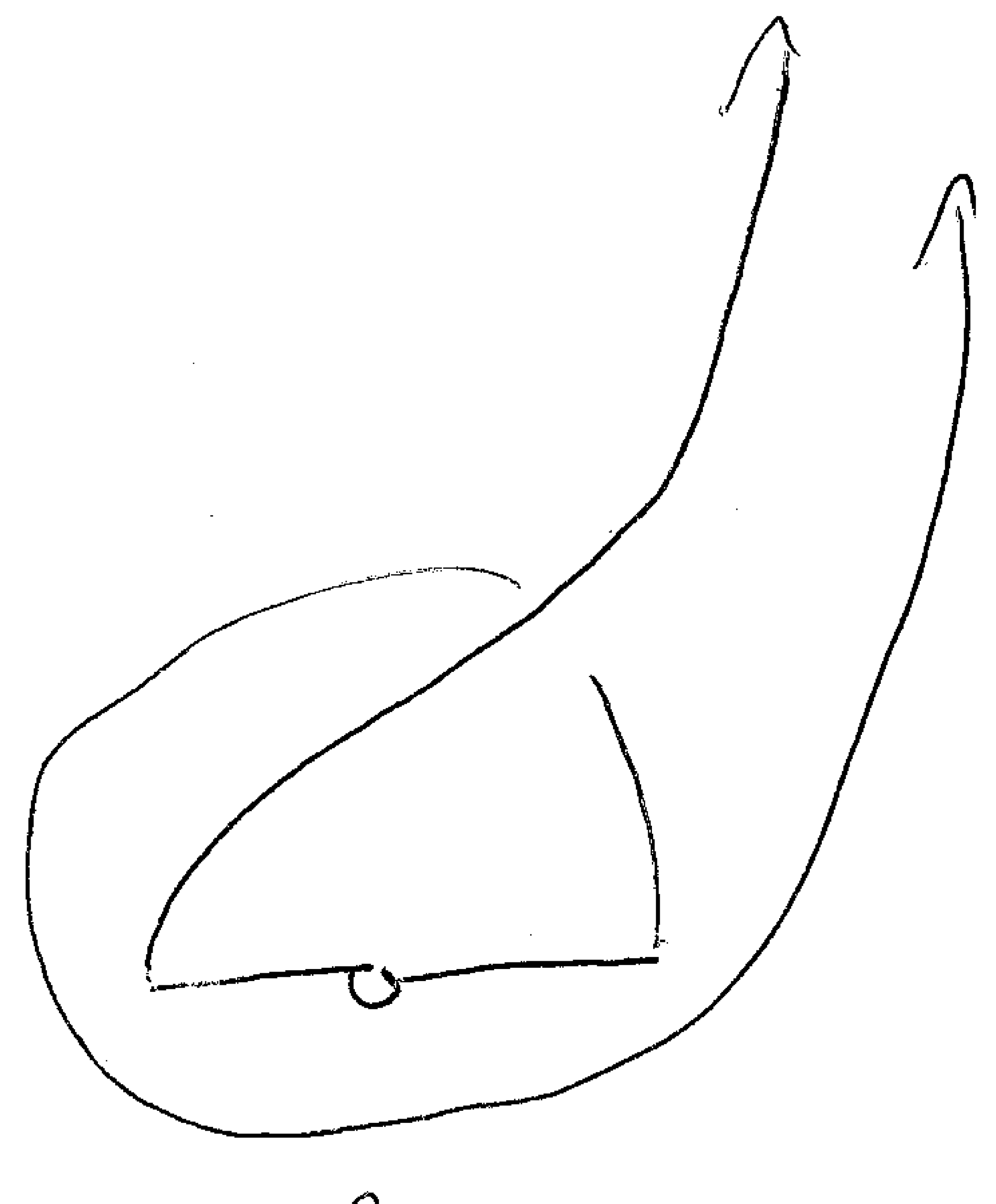


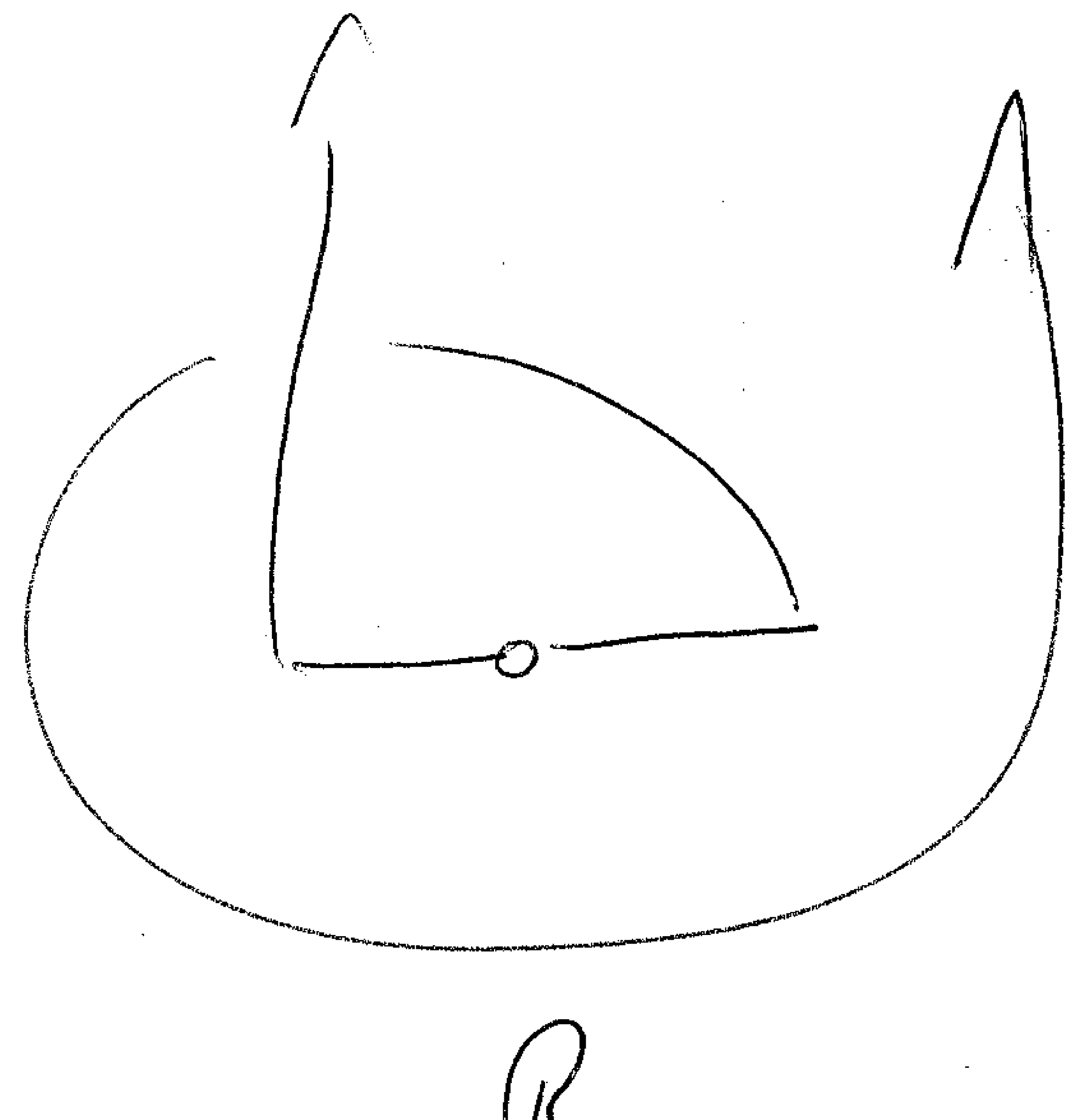
$\Leftarrow \lambda \gamma = \gamma \lambda$





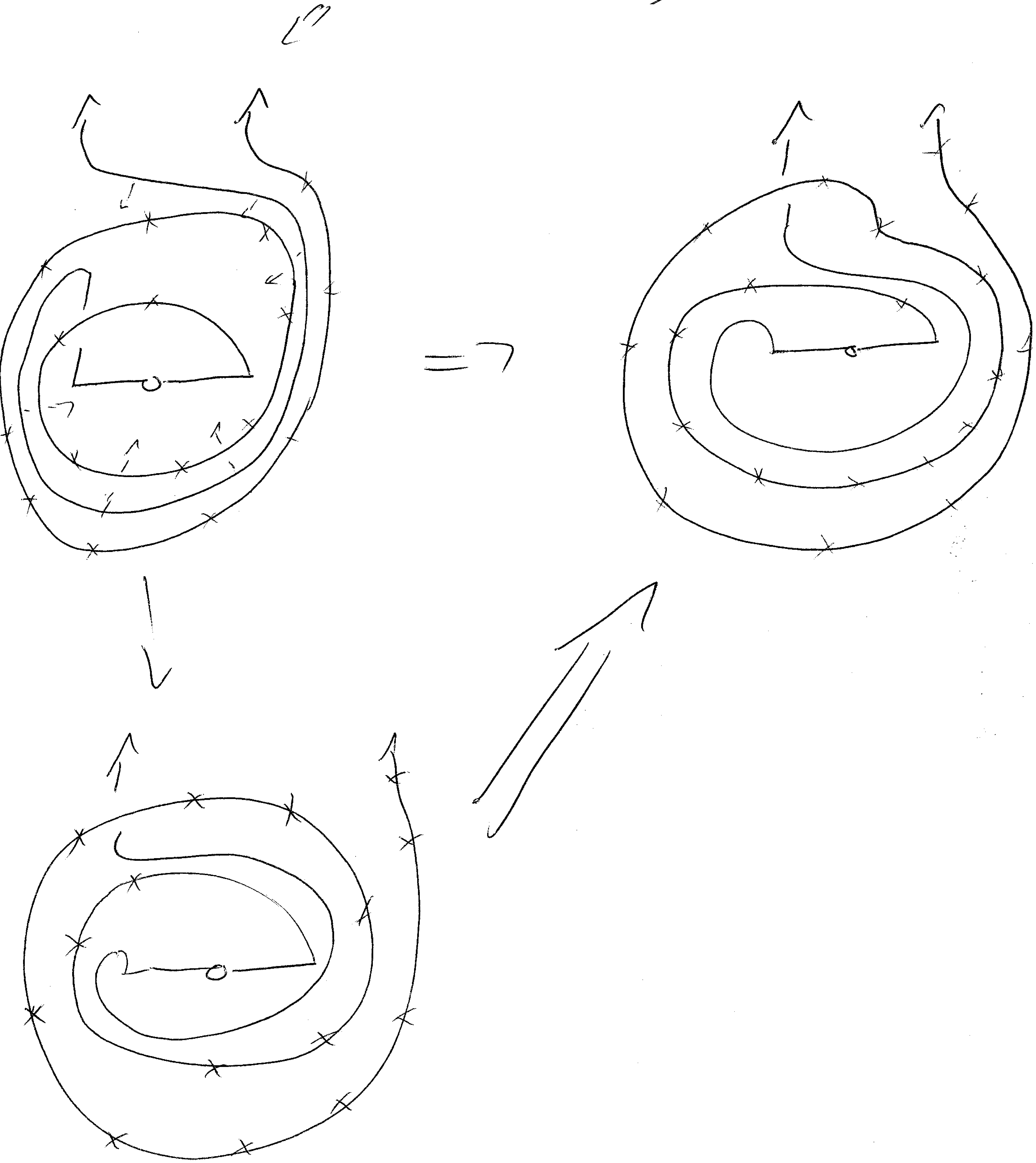
$\Downarrow$ $\Downarrow$



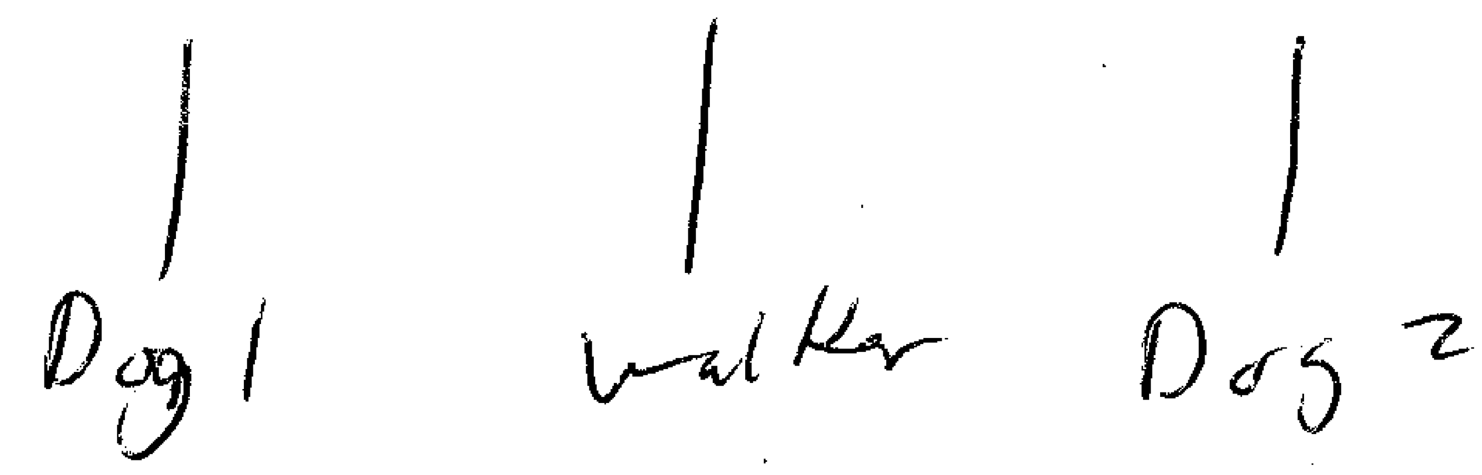


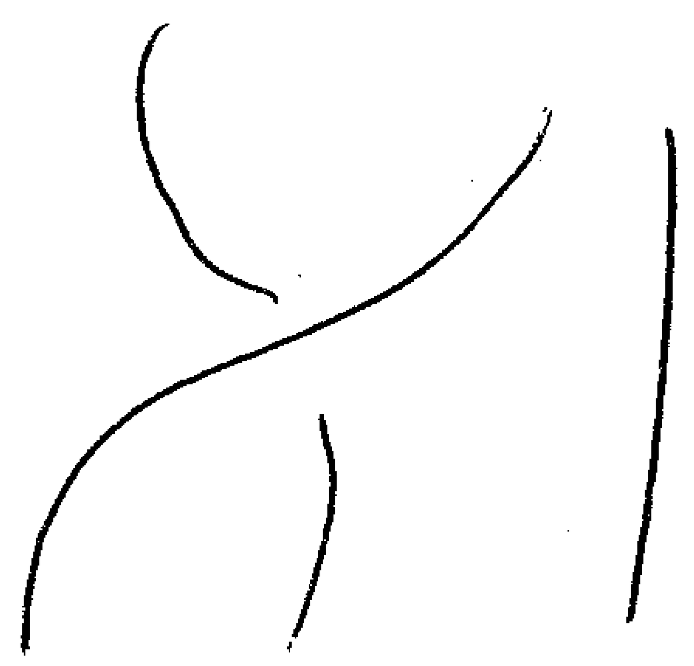
R R

$\rho\gamma = \gamma\rho \Rightarrow$

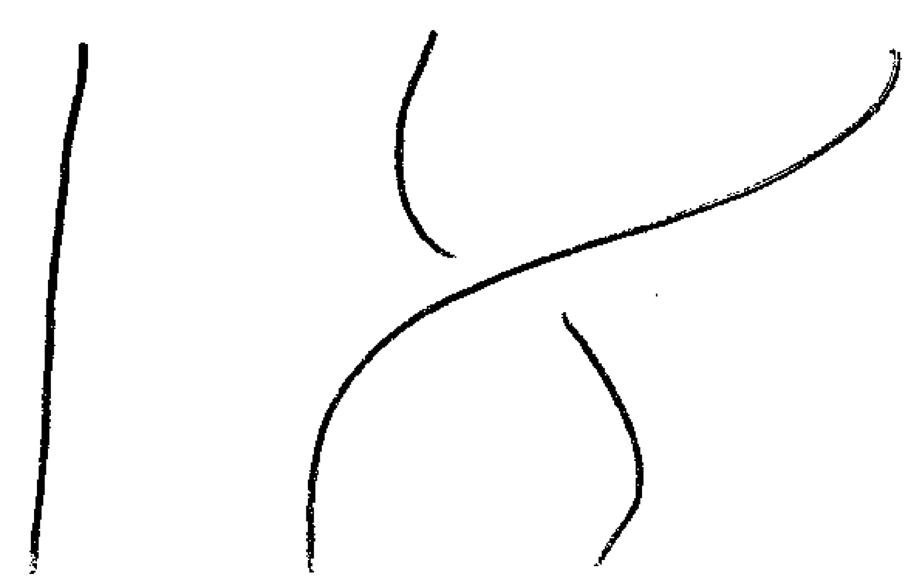


- Role of Braid Groups
  - To ensure that no generators or relations were left out
  - Relates tangled leashes to braids in B_3
    - B3 is the group that describes the 'hair' braids that can be made with three strings.
    - "The basic idea of B_n is to put a group structure on the set of all n stranded hair braids."
  - Explain sigma 1 and sigma 2 and the unbraid (| | |)





σ_1



σ_2

- *In General, The Braid Group on n string, B_n, is generated by elements*

Sigma 1, sigma 2,….. sigma (n-1)

Subject to the following relations:

Sigma i sigma j = sigma j sigma i

Sigma i sigma i+1 sigma i = sigma i+1 sigma i sigma i+1

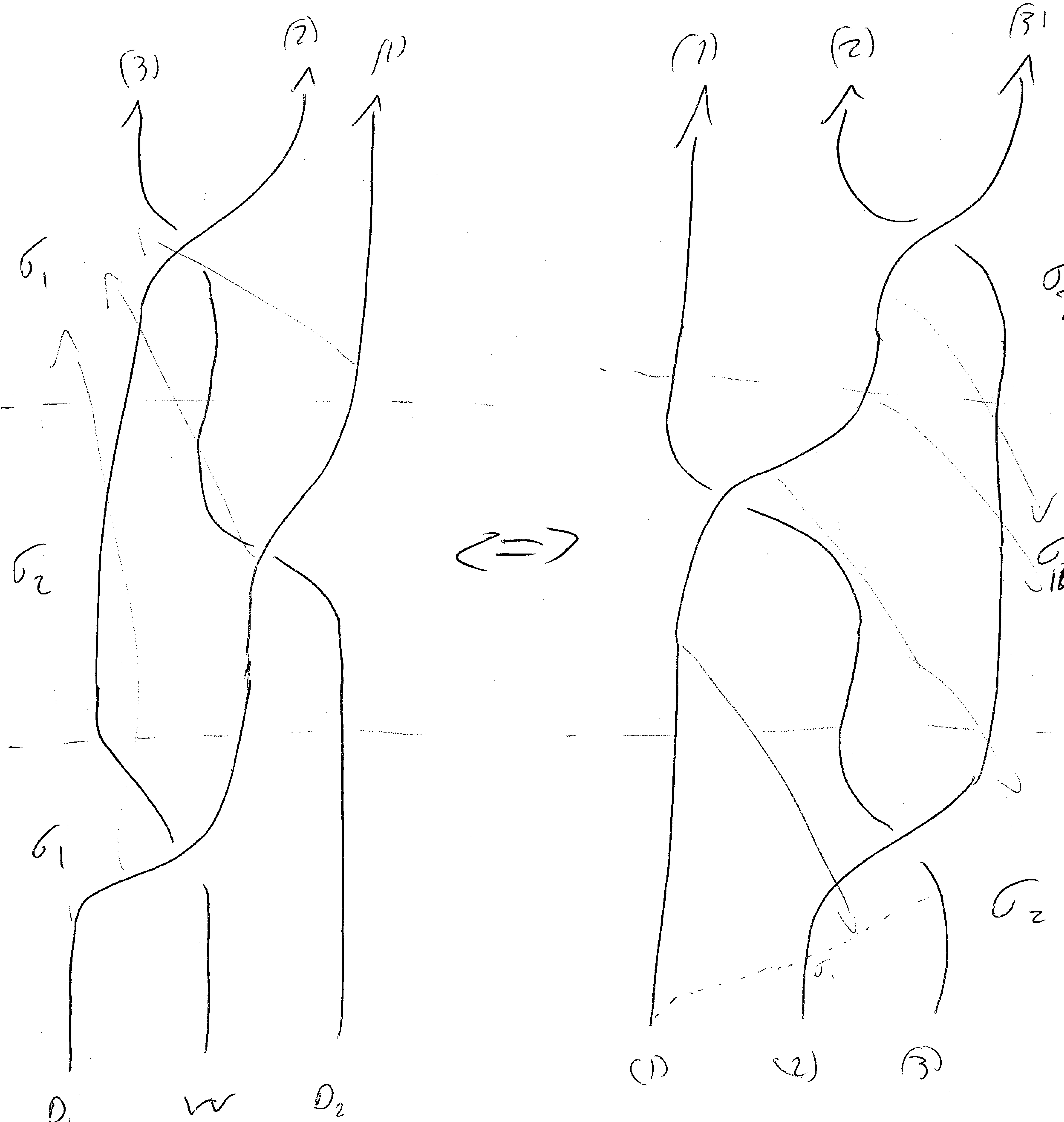
All he cares about is B_3 (Dog, Walker and Dog)

This is subject to the relation

Sigma 1 sigma 2 sigma 1 = sigma 2 sigma 1 sigma 2

Shown on next page

$$\Longleftarrow \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \quad \Longrightarrow$$



(1) (2) (3) NOTICE THAT MIDDLE STRING STAYS FIXED

$B_3^{(2)}$

- Theorem 2: The 2-Dog group is just $B_3^{(2)}$ as seen on previous page
    - B_3 where the second string (walker) stays fixed (b/c any move that is made by the dogs must keep you in the middle
    - Give simple setup of the proof (shown on the handout on next page)

Introduces Particle Dances

Theorem 2. *The 2-Dog Group, $\mathcal{D}_2$, defined intuitively in Section 1 above, is just $B_3^{(2)}$.* → B_3 where 2 stays fixed b/c any move must keep you in the middle

Remark. We'll also see that $\mathcal{W}$, the walker subgroup, has index 2 in $\mathcal{D}_2$. Therefore, half of the entanglements the dogs perform can be resolved by moves executed by the walker. In fact, all of the dogs' basic moves except T can be written in terms of the walker's generating moves. T is a representative for the non-identity coset of $\mathcal{W}$ in $\mathcal{D}_2$.

Proof. We first introduce an alternative interpretation of braids that will be helpful in the current context. This is the notion of *particle dances.* (See [Rolf, p. 78].) Suppose we have n particles in the complex plane, initially located at the points $1, \dots, n$, and that these particles begin to *dance* continuously in the plane, so that $\gamma_i(t) \in \mathbb{C}$ denotes the position at time t of the i-th particle, i.e, the particle that started at position i. If these dancing particles never collide and, when the music stops at time $t = 1$, they sit down in chairs located at $1, \dots, n$, then $\tau^i(t) = (\gamma_i(t), t)$ will be the i-th string of a braid as defined above. Conversely, given a braid, the projections of the string functions onto the xy-plane yield position functions for dancing particles.

The assumption that the dancing particles do not collide implies that, for each t, the ordered n-tuple $\Gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ is an element of the *configuration space* $K_n = (\mathbb{C} \times \dots \times \mathbb{C}) - \Delta$, where $\Delta = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j \text{ for some } i \neq j\}$. Since the particles are dancing continuously, this means that $\Gamma(t)$ is then a *path* in the space K_n that starts at the point $(1, \dots, n)$. Moreover, if all the dancing particles return to their starting points—i.e., we have a pure braid on n-strings—then $\Gamma(t)$ is a *loop* in the space K_n.

Fact. The group P_n of pure braids is isomorphic to $\pi_1(K_n)$, the fundamental group of K_n with basepoint $(1, \dots, n)$ [Rolf, p. 79].

Finally, $\mathcal{S}_n$ acts on K_n as usual by permuting coordinates, and $B_n = \pi_1(K_n')$, where $K_n' = K_n/\mathcal{S}_n$. In what follows, we will restrict ourselves to the case where $n = 3$.

From [Fad-Neu, p. 111], we see that $p\colon K_3 \to \mathbb{C}$ given by $p(z_1, z_2, z_3) = z_2$, the projection onto the second factor of K_3, is a locally trivial fibre bundle with fibre

$$F = p^{-1}(2) = ((\mathbb{C} - \{2\}) \times \{2\} \times (\mathbb{C} - \{2\})) - \Delta \approx \mathbb{C}^* \times \mathbb{C}^* - \Delta. \qquad (1)$$

The induced homomorphism p_* between fundamental groups sends a pure braid τ to the loop based at 2 gotten by considering only its second strand τ^2.

Now if we define $H \approx \mathbb{Z}/2$ to be the subgroup of $\mathcal{S}_3$ generated by $(1\ 3)$, the permutation of $\{1, 2, 3\}$ fixing 2, then H acts on K_3 by restriction and K_3/H is a covering space of K_3' and $\pi_1(K_3/H) = B_3^{(2)}$.

When restricted to the fibre, F, the H-action simply permutes its first and third factors. And, if we let H act trivially on $\mathbb{C}$, then the maps in the fibre bundle

$$F \xrightarrow{\iota} K_3 \xrightarrow{p} \mathbb{C} \qquad (2)$$

- Now just relate everything back to the dogs
- In conclusion, he found that he could reduce any number of dog moves down to the Group D2 (basically any number of T's and R's)

Do a Demonstraton