"Is it possible that the fundamental group of a manifold could be the identity, but that the manifold might not be homeomorphic to the three dimensional sphere?"

Henri Poincaré, 1904

Background:

- The poincaré conjecture is one of the seven Millennium Prize Problems established by the Clay Math Institute in 2000.
- The prize for solving one of these problems is $1,000,000.

Def: A space X is <u>simply connected</u> if it is path connected and its fundamental group is trivial.
(NO HOLES)

Def: A <u>manifold</u> is a topological space that is locally Euclidean
(around every point there is an open neighborhood that is homeomorphic to an open ball in $\mathbb{R}^n$).
example: a sphere.

Recall that a sphere (the skin of a 3-D ball) is two dimensional.

Def: The 3-sphere is the boundary of a 4 dimensional ball.

The Poincaré Conjecture (restated)
Every simply connected closed 3 dimensional manifold (i.e. a 3D object or the boundary of a 4D object) is homeomorphic to the 3 sphere.

This means that the 3 sphere is the only type of space with no holes.

The conjecture was attempted by many, and was finally proved in 2003 by a geometer, Grigori Perelman.

Key Players

Thurston
Hamilton
Perelman
Smale.

Smale proved true for dimensions >4.
(one & 2 dimensions were already known)
4 was proven later.

Thurston:

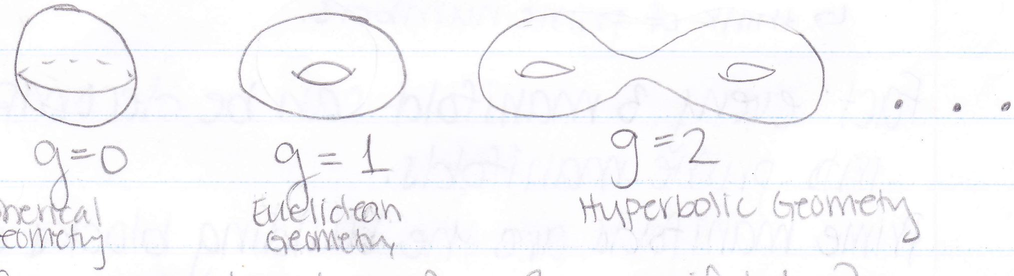
Def: A collection $\mathcal{A}$ of subsets of a space X is said to be a covering of X if the union of the elements of $\mathcal{A}$ is equal to X.

↳ $\mathcal{A}$ is an open covering of X if its elements are open subsets of X.

Def: A space X is compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Fact: we can list all possible smooth, compact orientable surfaces.

Def: A surface has genus $g \geq 0$, which is the number of holes it has



Q: Can we do this for 3-manifolds?

Thurston Geometrization Conjecture:

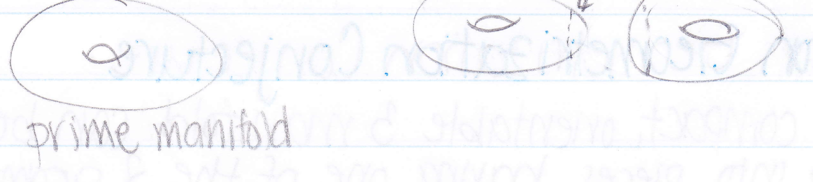
Every compact, orientable 3 manifold can be cut up into pieces having one of the 8 geometries.

Geometries

1. Euclidean
   Euclids 5th postulate holds
2. Hyperbolic
   only postulates 1-4 hold
3. Spherical
   only postulates 1-4 hold
4. $S^2 \times \mathbb{R}$
5. $H^2 \times \mathbb{R}$
6. universal cover of $\overline{SL_2 \mathbb{R}}$
7) Nil $\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$
8) Sol

Def: A prime manifold is a manifold that can be decomposed into a sphere & itself.
"cannot be split into more basic pieces"
↳ think of prime numbers.

Fact: every 3 manifold can be decomposed into prime manifolds.
Prime manifolds are the building blocks of manifolds.



Thurston Speculated:

prime manifolds can be taken from a pool of no more than 8 geometries.

Hamilton:

Developed the Ricci Flow Equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

expresses how geometry changes w/ time

"Ricci Tensor" a measure of curvature.

Def: curvature is an object that measures the deviation of the sum of int. angles of a triangle from 180°

↳ the inverse of the radius of the circle that best fits the curve at that point.

ex: line c=0 circle c=1

To describe curvature in higher dimensions we need an array of numbers: The Ricci Tensor.

* The Ricciflow disperses curvature throughout the manifold: The manifold expands where there is negative curvature and contracts where there is positive curvature.

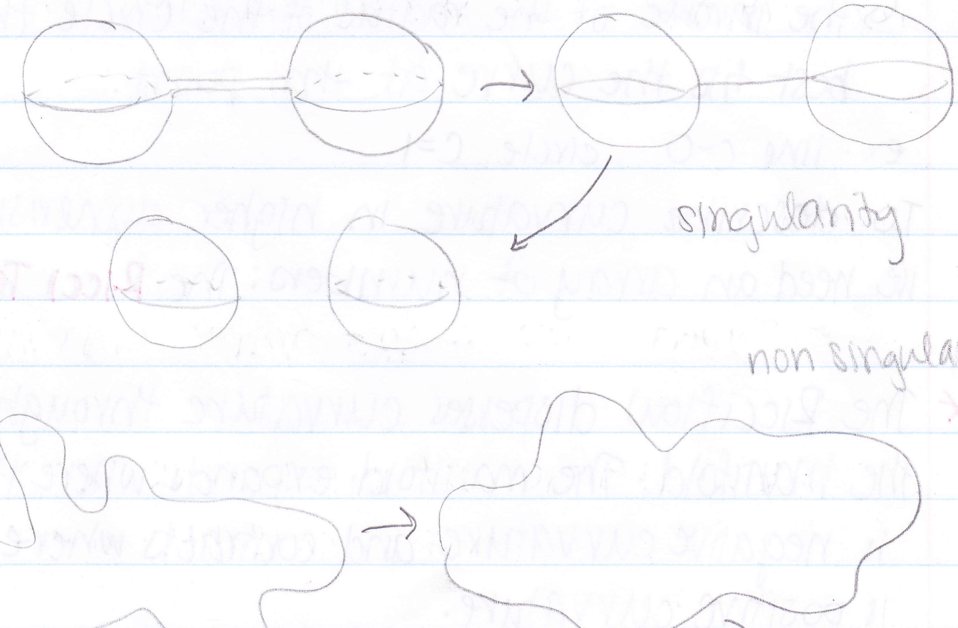
Analogue of Fourier Heat Equation.

Think of the Ricci Flow like botox: frown lines, forehead creases, crows feet disappear leaving the skin smooth. The lady is the same lady & the procedure used no "ripping, tearing!"

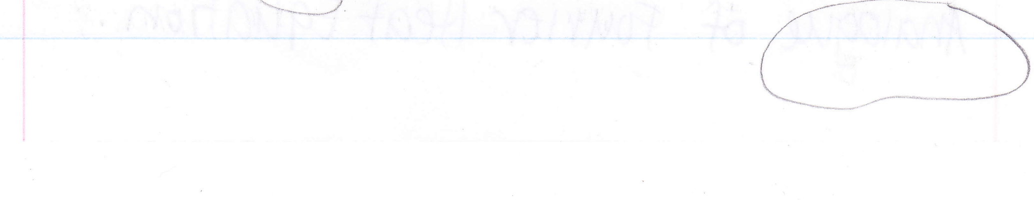
Hope: For a 3 manifold w/ finite fundamental group, under the Ricci flow, positive curvature would spread out (as $t \to \infty$) until the manifold has constant curvature.

Problem: <u>singularities</u>

curvature in different directions develops at different speeds.



non singularity



Hamilton speculated:

if there were singularities, and if one could find the structure of the singularity, one could cut off a piece of the manifold with the singularity (surgery), glue swaths onto both ends of the remaining manifold to cap the lessions, and continue with the Ricci flow.

If there are only a finite number of surgeries required in any finite amount of time, one could determine the topological structure of the manifold.

Hamilton listed all possible types of singularities.

Q: Does the ricci flow break a manifold into prime manifolds?

Perelman:

1) Developed tools to spot approaching singularities
2) Local Non-Collapsing Theorem:
   showed pathological singularities cannot occur.
3) invented methods to choose the right moment for surgery
4) Showed that only a finite # of surgeries are required.

- Let Ricci flow operate and observe until singularities occur.
- Do surgery
- Resume Ricci Flow
- Repeat unil finished

* The pieces removed during surgery are prime manifolds
* The remaining manifold is also a prime manifold.

The original manifold is made from these prime manifolds!
Geometrization Conjecture - proved!

* A simply connected, compact manifold that is deformed through Ricci flow & has all singularities removed through surgery will be a collection of spheres.
  → if we run time backwards and paste the spheres back together, the original manifold is itself just a sphere!

Poincaré Conjecture - proved!

2002, 2003 posted his proof online.
Fields Medal in 2006
↳ Did not accept

- $1,000,000 Millennium Prize
  ↳ Did not accept.

In 2003 he quit his job in academia.
He quit professional mathematics,

"As long as I was not conspicuous, I had a choice. Either to make some ugly thing or, if I didn't do this kind of thing, to be treated as a pet. Now, when I become a very conspicuous person, I cannot stay a pet and say nothing. This is why I had to quit."

He is currently jobless and lives at home with his mother.

Conclusion: If a 3 dimensional closed manifold has a trivial fundamental group (is simply connected) it is homeomorphic to the 3-sphere