Van Kampen's Theorem

- **Theorem I**f $X = A \cup B$, and A and B are open, and $A, B, A \cap B$ are path connected, then for $x_0 \in A \cap B$

$$\pi_1(X, x_0) \cong < \pi_1(A, x_0), \pi_1(B, x_0) | (i_A)_*(h) = (i_B)_*(h) \forall h \in \pi_1(X, x_0) >$$

  where i_A and i_B are inclusion maps.

- Remember that inclusion maps are of the form $i_A : A \cap B \to A$, since $A \cap B \subseteq A$, where $i_A(a) = a$.
- This theorem gives us the power to break up complicated shapes into simpler shapes whose fundamental groups we know, in order to find the fundamental group of the complicated shape.
- This is done by taking the generators of each fundamental and relations of each of the fundamental groups, and adding the relation $(i_A)_*(h) = (i_B)_*(h) \forall h \in \pi_1(X, x_0)$.
- This added relation ensures that paths in the intersection of A and B are identified.
- We proved that this is an isomorphism by showing
  - Homomorphism
    * So consider the diagram of inclusion maps. Note that both i_A and j_A are inclusion maps, it is only their domains and co-domains that change. So we will use the homomorphism $\theta : \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$ by

$$\theta = (j_A)_* * (j_B)_*$$

      This means that we do $(j_A)_*$ to elements of $\pi_1(A, x_0)$ and $(j_B)_*$ to elements of $\pi_1(B, x_0)$.
    * As an example, suppose we had a word in A and B, then

$$\theta(a_1 a_2 b_5 (a_3)^{-1}) = (j_A)_*(a_1)(j_A)_*(a_2)(j_B)_*(b_5)(j_A)_*(a_3^{-1})$$

    * So the first element is taken from A to $A \cap B$, etc
  - One to one
    * In this step we showed that elements contained in the intersection of A and elements in B that are the same element in the intersection become identified.
  - Onto
    * To show that the mapping is onto, we showed that every loop in X can be generated by combining loops in A and loops in B. To do so we broke up a loop in A and constructed it by using the fact that A and B are path connected and mapping to points on the loop from x_0 until we construct the whole loop.
- After proving the isomorphic relation, we looked at the examples of $S^1 \cup S^2$ and S_2, the 2-torus, and generated each of their fundamental groups.