# ANTENNAS

## Vector and Scalar Potentials

Maxwell's Equations

$$\nabla \times \mathbf{E} = -\,j\omega\mathbf{B} \qquad \text{(M1)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} \qquad \text{(M2)}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{(M3)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(M4)}$$

$$\mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathbf{B} = \mu\mathbf{H}$$

For a linear, homogeneous, isotropic medium $\mu$ and $\varepsilon$ are contant.

Since $\nabla \cdot \mathbf{B} = 0$, there exists a vector $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. $\mathbf{A}$ is called the *magnetic vector potential*. There are infinitely many vectors $\mathbf{A}$ that satisfy $\mathbf{B} = \nabla \times \mathbf{A}$ thus we later need to specify $\nabla \cdot \mathbf{A}$.

(M1) can be written as

$$\nabla \times \mathbf{E} = -j\omega\, \nabla \times \mathbf{A} \Rightarrow \nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0 \qquad (1)$$

$$\nabla \times \mathbf{Vector} = 0 \Rightarrow \mathbf{Vector} = \nabla(\text{some scalar}) \qquad (2)$$

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla\varphi \qquad \varphi : \text{scalar potential} \qquad (3)$$

$$\mathbf{E} = -\,j\omega\mathbf{A} - \nabla\varphi \qquad (4)$$

(M2) can be written as

$$\nabla \times \mathbf{B} = \mu\mathbf{J} + j\omega\mu\varepsilon\mathbf{E} \qquad (5)$$

$$\nabla \times \nabla \times \mathbf{A} = \mu\mathbf{J} + j\omega\varepsilon\mu\,(-j\omega\mathbf{A} - \nabla\varphi) \qquad (6)$$

$$-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu\mathbf{J} + \omega^2\mu\varepsilon\mathbf{A} - j\omega\varepsilon\mu\nabla\varphi \qquad (7)$$

To simplify the equation, choose $\nabla \cdot \mathbf{A}$ as

$$\nabla \cdot \mathbf{A} = - j\omega\mu\varepsilon\varphi \qquad (8)$$

This is called the *Lorentz Condition*. This leads to

$$\nabla^2 \mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = - \mu\mathbf{J} \qquad \textit{D'Alembert's Equation} \quad (9)$$

The problem now consists of finding the vector potential **A** due to a source **J**. From the knowledge of **A**, **E** and **H** can be determined.

Review spherical coordinates, gradient, divergence, curl, and laplacian in spherical coordinates. (Textbook, Appendix A pp. 689-691).

In spherical coordinates, the solution for the vector potential **A(r)** is given by

$$\mathbf{A(r)} = \mu \iiint_{V'} \frac{\mathbf{J(r')} e^{-j\beta|\mathbf{r}-\mathbf{r'}|}}{4\pi|\mathbf{r}-\mathbf{r'}|} \, dv' \qquad (10)$$

In the above equation, V' is the volume defining the source over which the integration is performed. **r** is the vector from the origin of the coordinate system, O, to the observer and **r'** is the vector from O to a source point within V'.

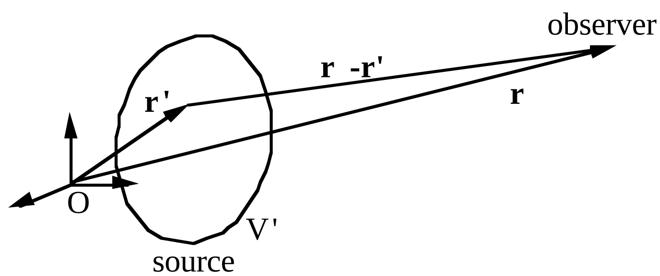


Figure 1

From **A**, use Maxwell's Equations to derive **E** and **H**.

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (11)$$

$$j\omega\varepsilon\mathbf{E} = \nabla \times \mathbf{H} \qquad (12)$$

**Hertzian Dipole**

Assume a source point infinitely small in size located at the origin of the coordinate system with elemental length dl, and driven by a current with strength $I_o$ in the +z direction. The equation for the current density of such a system is given by

$$\mathbf{J(r')} = \mathbf{i_z} I_o dl \delta(x')\delta(y')\delta(z') \tag{13}$$

Upon substitution in (10)

$$\mathbf{A(r)} = \mathbf{i_z} \frac{\mu I_o dl}{4\pi r} \exp(-j\beta r) \tag{14}$$

In spherical coordinates: $\mathbf{i_z} = \mathbf{i_r} \cos\theta - \mathbf{i_\theta} \sin\theta$

$$\mathbf{A(r)} = (\mathbf{i_r} \cos\theta - \mathbf{i_\theta} \sin\theta) \frac{\mu I_o dl}{4\pi r} \exp(-j\beta r) \tag{15}$$

$$A_r = \frac{\mu}{4\pi r} I_o dl \cos\theta \exp(-j\beta r) \tag{16}$$

$$A_\theta = - \frac{\mu}{4\pi} I_o dl \sin\theta \exp(-j\beta r) \tag{17}$$

Calculate E and H fields.

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \tag{18}$$

$$H_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] = 0 \tag{19}$$

$$H_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] = 0 \tag{20}$$

$$H_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \tag{21}$$

$$H_\phi = \frac{I_o dl}{4\pi r} \sin\theta \exp(-j\beta r)\, j\beta(1+ \frac{1}{j\beta r}) \tag{22}$$

Using $\mathbf{E} = \frac{1}{j\omega\varepsilon}\ \nabla \times \mathbf{H}$, we can derive the components of the electric field.

$$E_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi \sin\theta)\right]\frac{1}{j\omega\varepsilon} \tag{23}$$

$$E_r = \frac{2\cos\theta I_o dl}{4\pi r^2}\left\{e^{-j\beta r}\right\}j\beta\left(1+\frac{1}{j\beta r}\right)\frac{1}{j\omega\varepsilon} \tag{24}$$

$$E_\theta = \frac{1}{r}\left[-\frac{\partial}{\partial r}(rH_\phi)\right]\frac{1}{j\omega\varepsilon} \tag{25}$$

$$E_\theta = -\frac{j\beta I_o dl}{4\pi r}\sin\theta\left\{e^{-j\beta r}\right\}\left[-j\beta\left(1+\frac{1}{j\beta r}-\frac{1}{\beta^2 r^2}\right)\right]\frac{1}{j\omega\varepsilon} \tag{26}$$

In summary, the exact solutions for the fields of an infinitesimal antenna are given by

$$E_r = \frac{j\beta I_o dl}{4\pi r}e^{-j\beta r}2\cos\theta\left(\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)\sqrt{\frac{\mu}{\varepsilon}} \tag{27}$$

$$E_\theta = \frac{j\beta I_o dl}{4\pi r}e^{-j\beta r}\sin\theta\left(1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)\sqrt{\frac{\mu}{\varepsilon}} \tag{28}$$

$$E_\phi = 0$$

$$H_\phi = \frac{j\beta I_o dl}{4\pi r}e^{-j\beta r}\sin\theta\left(1+\frac{1}{j\beta r}\right) \tag{29}$$

$$H_r = H_\theta = 0$$

In most practical cases, the observer is located several wavelengths away from the source. This defines a *far field* region which is the region where the distance from the source to the observer is much larger than the wavelength $\lambda = \frac{2\pi}{\beta}$. In this case $r >> \lambda$ so that $\beta r >> 1$; consequently, the terms varying as $1/r^2$ and $1/r^3$ can be neglected. The far-field solutions for the infinitely small antenna thus become

$$E_r = 0 \tag{30}$$

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}\frac{j\beta I_o dl}{4\pi r}e^{-j\beta r}\sin\theta \tag{31}$$

$$H_\phi = \frac{j\beta I_o dl}{4\pi r} e^{-j\beta r} \sin\theta \qquad (32)$$

Note that the ratio $E_\theta/H_\phi$ is the characteristic impedance $\eta$ of the propagation medium. Over a small region, the far field solution is a plane-wave solution since the electric and magnetic fields are in phase, perpendicular to each other and their ratio is the intrinsic impedance $\eta$, and are perpendicular to the direction of propagation. However, unlike plane waves, the far field solution is a function of the elevation angle $\theta$, and does not have constant magnitude (1/r dependence).

## Radiation Patterns

The graph that describes the far-field strength versus the elevation angle at a fixed distance is called the *radiation pattern* of the antenna. In general, radiation patterns vary with $\theta$ and $\phi$. The distance from the dipole to a point on the radiation pattern is proportional to the field intensity or power density observed in that direction. Figure 2 shows the E-field and power density radiation patterns of a Hertzian dipole. As can be verified these patterns are based on the $\sin\theta$ and $\sin^2\theta$ dependence of the E-field and power density respectively.

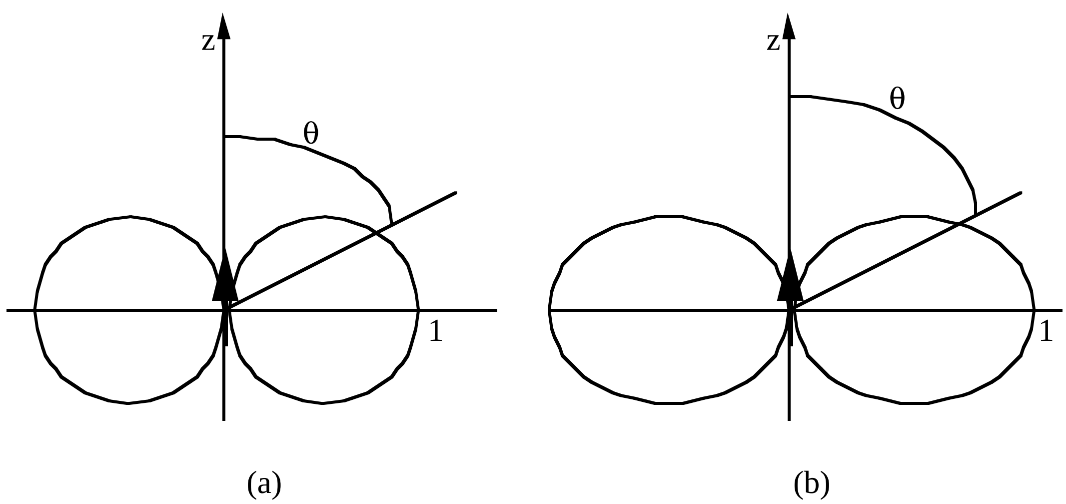


Figure 2. (a) Radiation pattern for E field (b) Radiation pattern for power density

## Time Average Power in Radiation Zone

In order to calculate the power radiated in the far field, we need to determine the time-average Poynting vector or power density <**P**>.

$$< \mathbf{P} > \;=\; \frac{1}{2}\text{Real}\left[\mathbf{E}\times\mathbf{H}^*\right] \;=\; \frac{\mathbf{i_r}}{2}\text{Real}\left[\sqrt{\frac{\mu}{\varepsilon}}\;|H_\phi|^2\right] \tag{33}$$

$$< \mathbf{P} > \;=\; \frac{\mathbf{i_r}\eta}{2}\text{Real}\left[\frac{\beta I_o dl}{4\pi r}\right]^2 \sin^2\theta \tag{34}$$

The total power radiated $P_T$ at a distance r is by definition obtained by integrating the Poynting vector over a sphere of radius r.

$$\text{Total Power } = P_T = \int_0^{2\pi}\int_0^{\pi} < \mathbf{P} > \cdot\, d\mathbf{S} \tag{35}$$

d**S** is the elemental surface of radius r and is given by

$$d\mathbf{S} = \mathbf{i_r} r^2 \sin\theta \; d\theta \; d\phi \tag{36}$$

so that

$$P_T \;=\; \int_0^{2\pi}\int_0^{\pi} \frac{\eta}{2}\left(\frac{\beta I_o dl}{4\pi r}\right)^2 r^2 \sin^3\theta \; d\theta \; d\phi \tag{37}$$

$$P_T \;=\; \frac{\eta}{2}\left|\frac{\beta I_o dl}{4\pi}\right|^2 \; 2\pi \int_0^{\pi} \sin^3\theta \; d\theta \tag{38}$$

$$P_T \;=\; \frac{4\pi\eta}{3}\left|\frac{\beta I_o dl}{4\pi}\right|^2 \;=\; \pi\frac{\eta}{3}\left|\frac{dl}{\lambda}\right|^2 |I_o|^2 \tag{39}$$

The ***directive gain*** is a figure of merit defined as

$$\text{Directive Gain} = \frac{\text{Poynting power density}}{\text{AveragePoynting power density over area of sphere of radius r}}$$

or

$$\text{Directive Gain} \;=\; \frac{< \mathbf{P} >}{P_T / 4\pi r^2} \tag{40}$$

For an infinitesimal antenna, we get

$$\text{Directive Gain} = \frac{\dfrac{\eta}{2}\left|\dfrac{\beta I_o dl}{4\pi r}\right|^2 \sin^2\theta}{\dfrac{4\pi\eta}{3}\left|\dfrac{\beta I_o dl}{4\pi}\right|^2} 4\pi r^2 = \frac{3}{2}\sin^2\theta \qquad (41)$$

The ***directivity*** is the directive gain in the direction of its maximum value. For an infinitesimal antenna, the direction of maximum value is for $\theta = \pi/2$ and the directivity is 1.5.

**Radiation Resistance**

The ***radiation resistance*** of an antenna is defined as the value of the resistor that would dissipate an equal amount of power than the power radiated for the same value of current. Using

$$P_T = \frac{1}{2} R_{rad} I_o^2 \qquad (42)$$

We get

$$R_{rad} = \frac{2P_T}{I_o^2} \qquad (43)$$

For an infinitesimal antenna, we get

$$R_{rad} = \frac{2}{I_o^2}\left(\frac{4\pi}{3}\right)\eta\left|\frac{\beta I_o dl}{4\pi}\right|^2 \qquad (44)$$

Using $\beta = \dfrac{2\pi}{\lambda}$, and $\eta = 120\pi$ ohms, we obtain

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 \ \Omega \qquad (45)$$