The Continuity Equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j}$$

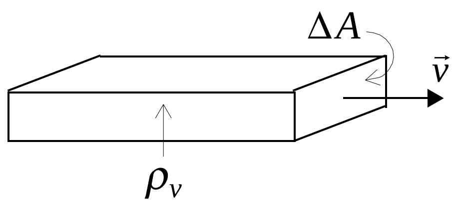
Current Density and the Continuity Equation

Current is motion of charges. In current theory, we have defined:

$$I = \frac{dQ}{dt} \quad (amps),$$

where I is the total current **through** a certain area (or device) and $\frac{dQ}{dt}$ is the rate of flow of charges through this area. (Area is "open" area.)

In EM, we are often interested in events at a point. Thus, we need current density (current per unit area: $amps/m^2$):



$dA \perp$ to $\vec{v}$

The total charge that flows **through** the area dA in time Δt is that in a box of length $\Delta t \cdot v$. The charge is then $\Delta Q = \rho_v v \Delta t \Delta A$, where ρ_v is the charge density , and A is the cross-sectional area.

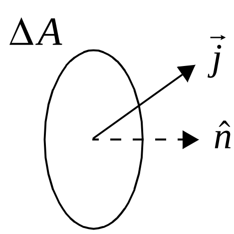
Therefore, $\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v v \Delta A$ and, thus, $j = \frac{\Delta I}{\Delta A} = \rho_v v$.

Moreover, j is in the direction of v.

Therefore, $\boxed{\vec{j} = \rho_v \vec{v}}$

If ρ_v is due to N electrons per unit volume, then $\rho_v = -Ne$. Then $\vec{j} = -Ne\vec{v}$

Now, if we have a surface not $\perp$ to $\vec{v}$ and $\vec{j}$,



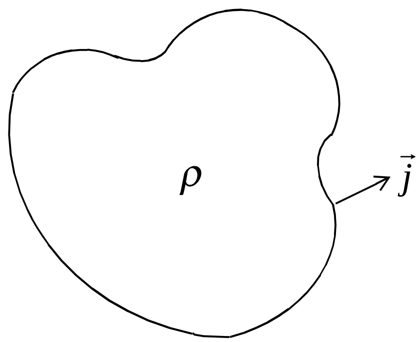
Only the component of $\vec{j}$ $\perp$ to the surface contributes to the flow of charge across the surface.

Therefore, $\Delta I = \vec{j} \cdot \Delta \vec{A}$

$$\boxed{\begin{aligned} dI &= \vec{j} \cdot d\vec{A} \\ I &= \iint_s \vec{j} \cdot d\vec{A} \end{aligned}}$$

[If $\vec{j}$ is along the normal and is constant, then $I = jA$.]

Now consider a **closed** surface S enclosing the volume V. Charges are flowing out of it. Therefore, there is current density $\vec{j}$ leaving the surface:



Total current flowing out of surface: $I = \iint_S \vec{j} \cdot d\vec{S}$

This must equal the rate of decrease of total charge inside. Thus:

$$I = -\frac{dq}{dt} = \iint_S \vec{j} \cdot d\vec{S}$$

But $q = \iiint_v \rho \, dv$

Therefore, $\iiint_v \frac{\partial \rho}{\partial t} dv = -\iint_S \vec{j} \cdot d\vec{S}$

Now we apply the Divergence Theorem:

$$\iint_S \vec{j} \cdot d\vec{S} = \iiint_v \nabla \cdot \vec{j} \, dv$$

Therefore, $\iiint_v \frac{\partial \rho}{\partial t} dv = -\iiint_v \nabla \cdot \vec{j} \, dv$

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j}}$$ **Continuity Equation**