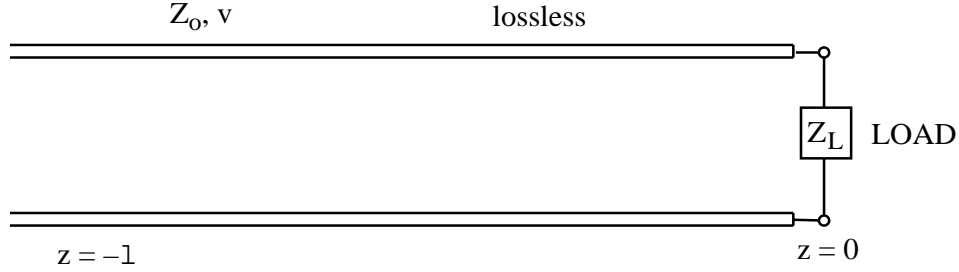


## Terminated Uniform Lossless Transmission Lines



Consider a lossless transmission line terminated in a load of impedance  $Z_L$ . A wave traveling to the right will be reflected at the termination. In general, there will be both positive going and negative going waves on the line. Hence,

$$\tilde{V}(z) = V_0 e^{-j\beta z} + V_1 e^{+j\beta z}. \quad (1)$$

Here,  $\gamma = j\beta$ ,  $\alpha = 0$ , because of no loss. The corresponding current, as in (5.32), is

$$\tilde{I}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \frac{V_1}{Z_0} e^{+j\beta z}, \quad (2)$$

where  $Z_0 = \sqrt{\frac{L}{C}}$  and  $\beta = \omega\sqrt{LC}$  for a lossless line.

At  $z = 0$ ,

$$\frac{\tilde{V}(z=0)}{\tilde{I}(z=0)} = Z_L = \frac{V_0 + V_1}{V_0 - V_1} Z_0. \quad (3)$$

We can solve for  $V_1$  in terms of  $V_0$ , i.e.

$$V_1 = \frac{Z_L - Z_0}{Z_L + Z_0} V_0. \quad (4)$$

If we define

$$\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0}, \quad (5)$$

then  $V_1 = \rho_v V_0$ , and Equation (1) becomes

$$\tilde{V}(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \quad (6)$$

In the above,  $\rho_v$  is the ratio of the negative going voltage amplitude to the positive going voltage amplitude at  $z = 0$ , and it is known as the **voltage reflection coefficient**

The **current reflection coefficient** is defined as the ratio of the negative going current to the positive going current at  $z = 0$ , and it is

$$\rho_i = \frac{I_1}{I_0} = -\frac{V_1}{V_0} = -\rho_v. \quad (7)$$

The current can be written as

$$\tilde{I}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \rho_v \frac{V_0}{Z_0} e^{j\beta z}. \quad (8)$$

The voltage and current in (6) and (8) are not constants of position. We can define a **generalized impedance** at position  $z$  to be

$$Z(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \frac{e^{-j\beta z} + \rho_v e^{j\beta z}}{e^{-j\beta z} - \rho_v e^{j\beta z}}. \quad (9)$$

At  $z = -l$ , this becomes

$$Z(-l) = Z_0 \frac{e^{j\beta l} + \rho_v e^{-j\beta l}}{e^{j\beta l} - \rho_v e^{-j\beta l}}. \quad (10)$$

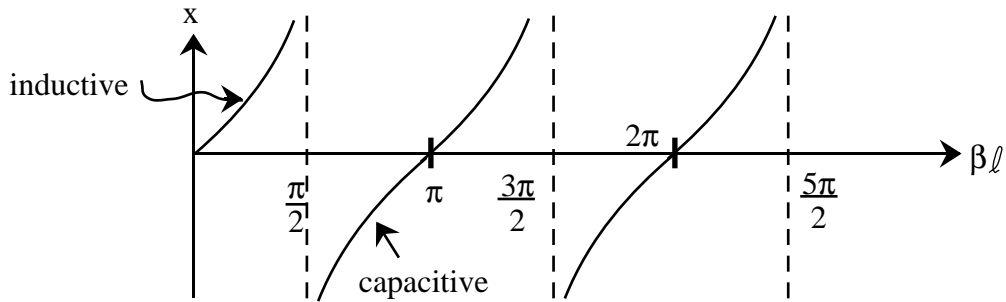
With  $\rho_v$  defined by (5), we can substitute it into (10) to give after some simplifications,

$$Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}. \quad (11)$$

### Shorted Terminations

If  $Z_L$  is a short, or  $Z_L = 0$ , then,

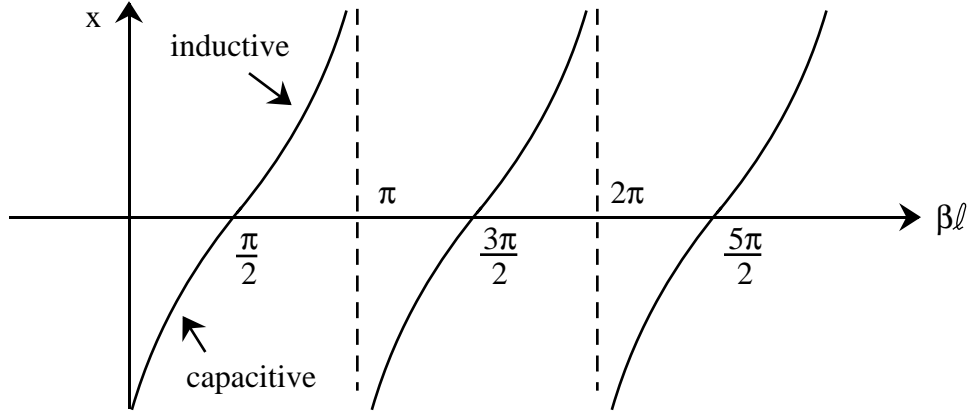
$$Z(-l) = jZ_0 \tan \beta l = jX. \quad (12)$$



### Open-Circuit Terminations

If  $Z_L$  is an open circuit,  $Z_L = \infty$ , then

$$Z(-l) = -jZ_0 \cot \beta l = jX. \quad (13)$$



### Standing Waves on a Lossless Transmission Line

The positive going wave in Equation (6) is

$$V_+(z) = V_0 e^{-j\beta z}, \quad (14)$$

and the negative going wave in Equation (6) is

$$V_-(z) = \rho_v V_0 e^{+j\beta z}. \quad (15)$$

We can define a **generalized reflection coefficient** to be the ratio of  $V_+(z)$  to  $V_-(z)$  at position  $z$ . Hence,

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \rho_v e^{2j\beta z}. \quad (16)$$

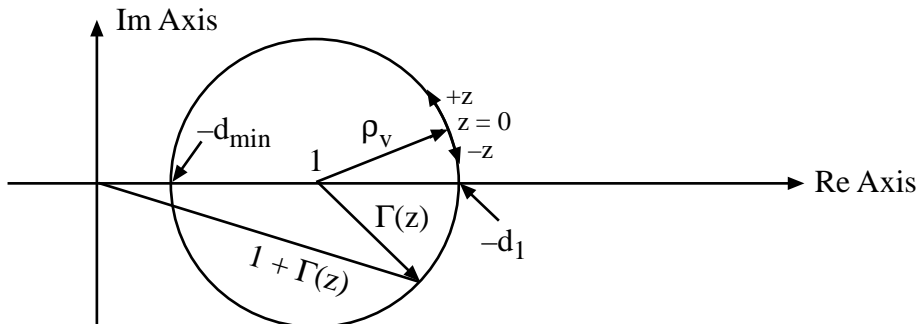
Hence,

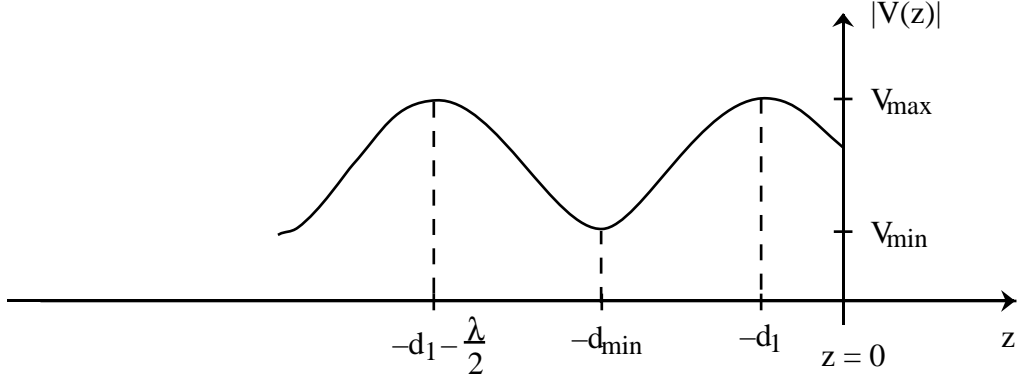
$$V(z) = V_0 e^{-j\beta z} [1 + \Gamma(z)]. \quad (17)$$

The magnitude of  $V(z)$  is then

$$|V(z)| = |V_0| |1 + \Gamma(z)|. \quad (18)$$

A plot of  $|V(z)|$  is as shown.





We can use the triangular inequality and show that

$$|V_0|(1 - |\Gamma(z)|) \leq |V(z)| \leq |V_0|(1 + |\Gamma(z)|). \quad (19)$$

From (16),  $|\Gamma(z)| = |\rho_v|$ , hence (19) becomes,

$$|V_0|(1 - |\rho_v|) \leq |V(z)| \leq |V_0|(1 + |\rho_v|). \quad (20)$$

The voltage standing wave ratio is defined to be  $V_{max}/V_{min}$ , and from (20), it is

$$\text{VSWR} = \frac{1 + |\rho_v|}{1 - |\rho_v|}. \quad (21)$$

If  $\rho_v = 0$ , then  $\text{VSWR} = 1$ , and we have no reflected wave. We say that the load is matched to the transmission line. Note that  $\rho_v = 0$  when  $Z_L = Z_0$ .

If  $|\rho_v| = 1$ , then  $\text{VSWR} = \infty$ , and we have a badly matched transmission line. In a passive load,

$$0 \leq |\rho_v| \leq 1. \quad (22)$$

$|\rho_v| = 1$  only when  $Z_L = 0$ , or  $Z_L = \infty$  according to Equation (5). Hence,

$$1 \leq \text{VSWR} < \infty. \quad (23)$$

VSWR is an indicator of how well a load is being matched to the transmission line. We can solve (21) for  $|\rho_v|$  in terms of VSWR, i.e.

$$|\rho_v| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1}. \quad (24)$$

Therefore, given the measurement of  $\text{VSWR}$  on a terminated transmission line, we can deduce the magnitude of  $\rho_v$ . Furthermore, if we know the phase of  $\rho_v$ , we would be able to derive  $Z_L$  from (5), or

$$Z_L = Z_0 \frac{1 + \rho_v}{1 - \rho_v}, \quad (25)$$

or

$$Z_L = Z_0 \frac{1 + |\rho_v| e^{j\theta_v}}{1 - |\rho_v| e^{j\theta_v}}, \quad (26)$$

where

$$\rho_v = |\rho_v| e^{j\theta_v}. \quad (27)$$

### Determining $\theta_v$ from $|V(z)|$

$\theta_v$  can be determined from the voltage standing wave measured. The voltage standing wave pattern is proportional to  $|1 + \Gamma(z)|$ , but  $\Gamma(z)$  is related to  $\rho_v$  as

$$\Gamma(z) = \rho_v e^{2j\beta z}. \quad (28)$$

Writing the polar representation of  $\rho_v$ , we have,

$$\Gamma(z) = |\rho_v| e^{j(2\beta z + \theta_v)}. \quad (29)$$

However, we know that the first minimum value of  $V(z)$  occurs when  $\Gamma(z)$  is purely negative, or the phase of  $\Gamma(z)$  is  $-\pi$ . This occurs at  $z = -d_{min}$  first. In other words,

$$-2\beta d_{min} + \theta_v = -\pi. \quad (30)$$

Since  $d_{min}$  can be obtained from the voltage standing wave pattern measurement, and that  $\beta = 2\pi/\lambda$ , we deduce that

$$\theta_v = -\pi + \frac{4\pi}{\lambda} d_{min}. \quad (31)$$

### Transmission Coefficients

It is sometimes useful to define a transmission coefficient on a transmission line. The transmission coefficient may be defined as the ratio of the voltage on the load to the amplitude of the incident voltage. Since

$$V(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \quad (32)$$

The voltage at the load is  $V(z = 0)$ , and it is given by

$$V(0) = V_0(1 + \rho_v). \quad (33)$$

Since the amplitude of the incident voltage is  $V_0$ , we have

$$\tau_v = \frac{V(0)}{V_0} = 1 + \rho_v = \frac{2Z_L}{Z_L + Z_0}. \quad (34)$$