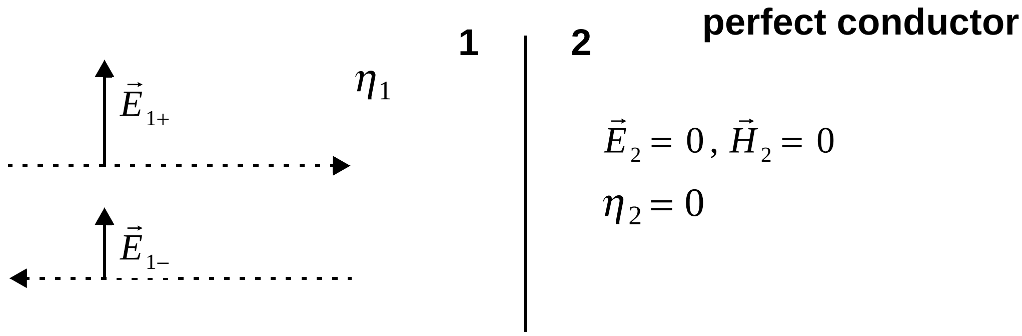
Plane Wave Reflections at a Conductor

Power and reflection coefficient at dielctric boundary

$$\eta = \eta_1 \frac{\eta_2 \cos kd + j\eta_1 \sin kd}{\eta_1 \cos kd + j\eta_2 \sin kd}$$

Normal Incidence Plane Wave Reflection at Perfect Conductor



At the boundary, since $\vec{E}_2$ and $\vec{H}_2$ are both 0, then:

$$\vec{E}_{tot} = \vec{E}_{1+} + \vec{E}_{1-} = 0 \qquad E \text{ tangential} = 0 \text{ (no charge)}$$

Solution exists for $\vec{E}_{1+} = -\vec{E}_{1-} = E$

Then, $\vec{E}_{tot}(z) = \vec{E}_{1+}\left(e^{-j\beta_1 z} - e^{+j\beta_2 z}\right)\hat{e}_x = -2\,j\,E\sin\beta_1 z\,\hat{e}_x$

This is our old friend, the standing wave! Let's check ρ and τ.

Recall that $\rho = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1$, since $\eta_2 = 0$ and $\tau = \dfrac{2\eta_2}{\eta_2 + \eta_1} = 0$.

The total field is: $\vec{E}_{tot}(z,t) = \mathrm{Re}\left[e^{j\omega}\vec{E}_{tot}(z)\right] = 2\sin\left(\beta_1 z\right)\sin\omega t\,\hat{e}_x$.

What about the **magnetic field**?

$$\vec{H}_{tot}(z) = \left(\frac{E_+}{\eta_1}e^{-j\beta_1 z} + \frac{E_-}{-\eta_1}e^{+j\beta_2 z}\right)\hat{e}_y$$

But the boundary conditions on E have that $E_+ = -E_-$.

Then:

$$\vec{H}_{tot}(z) = \frac{E_+}{\eta_1}\left(e^{-j\beta_1 z} + e^{+j\beta_2 z}\right)\hat{e}_y = 2\frac{E_+}{\eta_1}\cos\beta z\,\hat{e}_y$$

or, in the time domain representation:

$$\vec{H}_{tot}(z,t) = 2\frac{E_+}{\eta_1}\cos\beta z\,\cos\omega t\,\hat{e}_y$$

Observations:

1. The amplitude of the total electric field is zero at the surface of the conductor. (For all time!)
2. The maximum amplitude of the standing wave is double that of the travelling wave. The maximum occurs at specific locations, namely:

$$z = \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots \text{ and, at specific times, } \omega t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots.$$

   The location of the maximum coincides with the maximum in constructive interference.
3. There are points in front of the metal reflector where the total electric field is always zero. These are the points of maximum destructive interference:

$$z = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$$

4. Location of nulls and peaks do not change in time.
5. The electric and magnetic fields are $90°$ out of phase in a standing wave! This leads to no net power flow.
6. New $\rho_{\vec{H}}$ and $\tau_{\vec{H}}$ for $\vec{H}$ field:

$$\rho_{\vec{H}} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} = -\rho \text{ and } \tau_{\vec{H}} = \frac{2\eta_1}{\eta_1 - \eta_2} = \frac{\eta_2}{\eta_1}\tau$$

Example

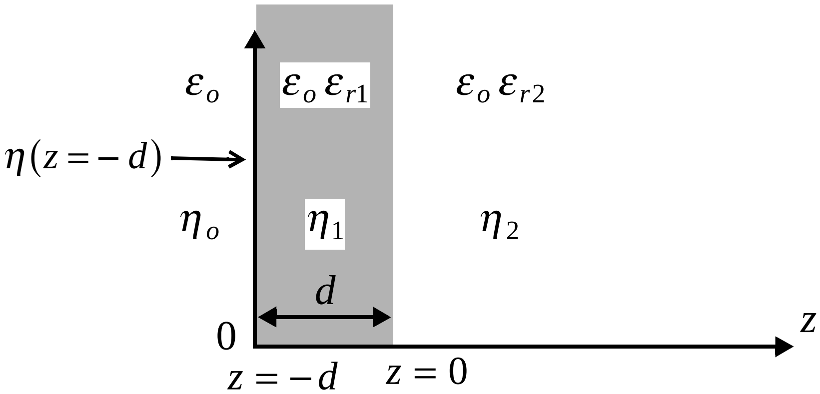
What is the Poynting vector for this wave?

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left[\vec{E}(z) \times \vec{H}^*(z)\right]$$

$$= \frac{1}{2}\mathrm{Re}\left(-2\,j\,E\sin\beta_1 z\,\hat{e}_x \times \frac{2\,E}{\eta_1}\cos\beta_1 z\,\hat{e}_y\right) = 0 \quad \text{(Why?!)}$$

Reflections at Multiple Intefaces

We previously calculated the impedance at a distance $-z$ from an interface. Suppose we have a dielectric ($\varepsilon = \varepsilon_o \varepsilon_r$) slab of thickness d in air ($\varepsilon = \varepsilon_o$) and a plane wave incident normal to the surface.



The impedance at $z = -d$ is given by:

$$\eta = \eta_1 \frac{\eta_2 \cos kd + j\eta_1 \sin kd}{\eta_1 \cos kd + j\eta_2 \sin kd}$$

This is a complicated expression that has some very interesting special cases.

To get one of these cases, note that if $kd = \frac{\pi}{2}$, then the cosines are zero and $\eta = \eta_1 \frac{j\eta_1}{j\eta_2} = \frac{\eta_1^2}{\eta_2}$.

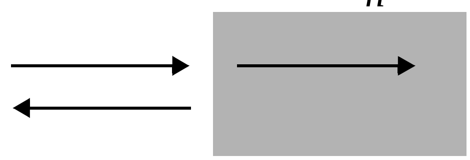
The importance of this result is that if we can find an η_1, such that $\eta_o = \frac{\eta_1^2}{\eta_2}$, then there will be no reflections from the boundary at $z = -d$. For example, select $\eta_1 = \sqrt{\eta_o \eta_2}$. Then $\rho = \frac{\eta - \eta_o}{\eta + \eta_o} = 0$ because $\eta = \eta_o$.

Examples of Reflection and Transmission Coefficients

Suppose we have a very thick slab of a semiconductor with $n = 3.5$. Let us look at the reflection at the interface with air ($\eta_{air} \approx \eta_o$).

First from air to semiconductor:

$$\eta_1 = \eta_o \qquad\qquad \eta_2 = \frac{\eta_o}{n}$$



Recall that $\eta = \dfrac{\eta_o}{n}$.

$$\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\dfrac{\eta_o}{n} - \eta_o}{\dfrac{\eta_o}{n} + \eta_o} = \frac{\dfrac{1}{n} - 1}{\dfrac{1}{n} + 1}$$

$$= \frac{1 - n}{1 + n} = \frac{-2.5}{4.5} = -0.55555... = \frac{E_{1-}}{E_{1+}}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{\dfrac{2\eta_o}{n}}{\dfrac{\eta_o}{n} + \eta_o} = \frac{\dfrac{2}{n}}{\dfrac{1}{n} + 1}$$

$$= \frac{2}{n+1} = \frac{2}{4.5} = 0.444444... = \frac{E_{2+}}{E_{1+}}$$

$$1 + \rho = \tau \Rightarrow 1 - 0.55555... = 0.444444...$$

Check power: $P_{incident} = \dfrac{E_{1+}^2}{2\eta_o}$

$$P_{reflected} = \frac{E_{1-}^2}{2\eta_o} = P_{incident} \left|\rho\right|^2 = P_{incident}\, 0.30864$$

$$P_{transmitted} = \frac{E_{2+}^2}{2\eta_2} = \frac{\left(\tau\, E_{1+}\right)^2 n}{2\eta_o} = P_{incident}\, \tau^2 n$$

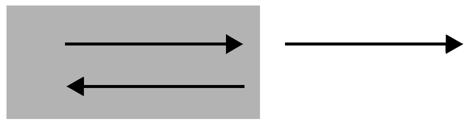
$$= P_{incident}\, 0.1975 \times 3.5 = P_{incident}\, 0.6913$$

or: $P_{incident} = P_{reflected} + P_{transmitted} = P_{incident}\,(0.6913+0.30864) = 0.99999$

That is, power is indeed conserved!

Now we will reverse the direction of the radiation.

$$\eta_1 = \frac{\eta_o}{n} \qquad\qquad \eta_2 = \eta_o$$



Recall again that $\eta = \dfrac{\eta_o}{n}$. This time we have:

$$\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_o - \frac{\eta_o}{n}}{\eta_o + \frac{\eta_o}{n}} = \frac{1-\frac{1}{n}}{1+\frac{1}{n}}$$

$$= \frac{n-1}{n+1} = \frac{2.5}{4.5} = 0.55555\ldots = \frac{E_{1-}}{E_{1+}}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\eta_o}{\eta_o + \frac{\eta_o}{n}} = \frac{2n}{1+\frac{1}{n}}$$

$$= \frac{2n}{1+n} = \frac{7}{4.5} = 1.555555\ldots = \frac{E_{2+}}{E_{1+}}$$

$$1+\rho = \tau \Rightarrow 1 + 0.55555\ldots = 1.55555\ldots$$

Check power: $P_{incident} = \dfrac{E_{1+}^2}{2\eta} = \dfrac{E_{1+}^2\, n}{2\eta_o} \Rightarrow \dfrac{E_{1+}^2}{2\eta_o} = \dfrac{P_{incident}}{n}$

Then:

$$P_{reflected} = \frac{E_{1-}^2}{2\eta} = P_{incident}\,|\rho|^2 = P_{incident}\,0.30864$$

$$P_{transmitted} = \frac{E_{2+}^2}{2\eta_o} = \frac{(\tau E_{1+})^2}{2\eta_o} = \frac{P_{incident}\,\tau^2}{n} = \frac{2.4197}{3.5}$$

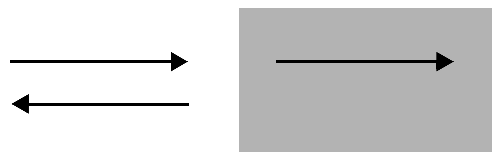
$$= P_{incident}\,0.6913$$

or: $P_{incident} = P_{reflected} + P_{transmitted} = P_{incident}\,(0.6913+0.30864) = 0.99999$

That is, power is again conserved!

Consider two limiting cases: Suppose we have a boundary between two dielectrics as above, except now let the dielectric constant of one be nearly infinite. What are the reflection coefficients?

$$\eta_1 = \eta_o \qquad \eta_2 = \frac{\eta_o}{n} \qquad n >> 1$$



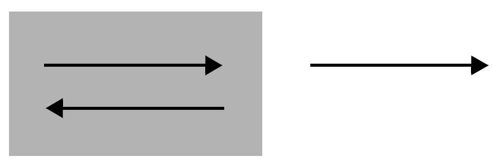
$$n >> 1, \eta_2 \approx 0 \qquad \rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \approx -1 \qquad 1 + \rho = \tau$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} \approx 0$$

Electric field flips over, like a dielectric metal interface. (short circuit)

Now the other case:

$$n >> 1 \qquad \eta_1 = \frac{\eta_o}{n} \qquad \eta_2 = \eta_o$$



$$n >> 1, \eta_1 \approx 0 \qquad \rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \approx 1 \qquad 1 + \rho = \tau$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} \approx 2$$

$$1 + \frac{E_{1-}}{E_{1+}} = \frac{E_{2+}}{E_{1+}} \Rightarrow E_{1+} + E_{1-} = E_{2+}$$