Plane Wave Reflections at Boundaries

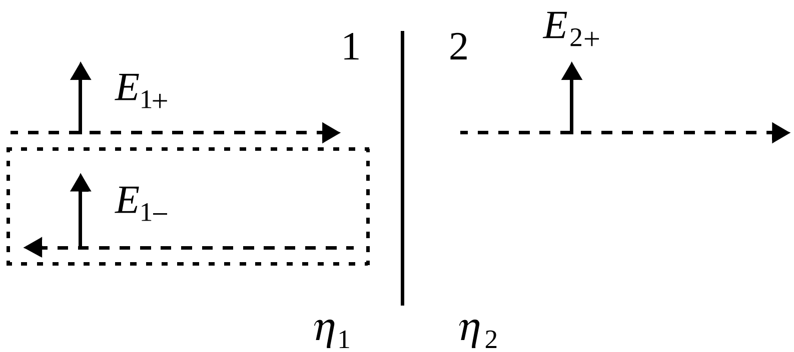
$$\mathbf{H}_y = \frac{\mathbf{E}_{x+}}{\eta} - \frac{\mathbf{E}_{x-}}{\eta}$$

Plane Wave Propagation, Reflections at Boundaries and Impedance

We established the relations among $\vec{E}$ and $\vec{H}$ for a plane wave propagating in the z direction earlier. We use these relations to consider reflection at a boundary. The relations are:

$$\mathbf{E}_x = E_+ e^{-jkz} + E_- e^{+jkz}$$
$$\mathbf{H}_y = \frac{\mathbf{E}_{x+}}{\eta} - \frac{\mathbf{E}_{x-}}{\eta}$$
$$k = \omega(\mu\,\varepsilon)^{1/2} = \frac{\omega}{v}$$
$$\eta = \left(\frac{\mu}{\varepsilon}\right)^{1/2}$$

Field Reflection Coefficient, ρ



At $z = 0$, we must apply the boundary conditions:

For the electric field: E tangential is continuous.

$$E_{1+} = E_{2+}$$

Now for the magnetic field:

$\frac{E_{1+}}{\eta} = \frac{E_{2+}}{\eta}$ (Remember that $\mathbf{H}_y = \frac{\mathbf{E}_x}{\eta}$, a plane wave)

We cannot satisfy both of these equations at the same time. We conclude that we must add a reflected field for a non-trivial solution.

Let's apply boundary conditions again with the addition of a reflected wave.

$$E_{1+}e^{-jkz} + E_{1-}e^{+jkz} = E_{2+}e^{-jkz} \text{ (drop propagator)}$$

and $$\frac{E_{1+} - E_{1-}}{\eta_1} = \frac{E_{2+}}{\eta_2},$$

or: $$E_{2+} = \frac{\left(E_{1+} - E_{1-}\right)\eta_2}{\eta_1}$$

Then $$E_{1+} + E_{1-} = \frac{\left(E_{1+} - E_{1-}\right)\eta_2}{\eta_1}$$

$$E_{1+}\left(1 - \frac{\eta_2}{\eta_1}\right) + E_{1-}\left(1 + \frac{\eta_2}{\eta_1}\right) = 0,$$

or: $$\frac{E_{1-}}{E_{1+}} = \rho = -\frac{1 - \frac{\eta_2}{\eta_1}}{1 + \frac{\eta_2}{\eta_1}}$$

$$\boxed{\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$

The transmission coefficient is defined as: $\tau = \frac{E_{2+}}{E_{1+}}$

From the first relations, we have: $E_{1-} = -E_{1+} + E_{2+}$.

Recall that: $$\frac{E_{1+} - E_{1-}}{\eta_1} = \frac{E_{2+}}{\eta_2}$$

So $$\frac{E_{1+} + E_{1+} - E_{2+}}{\eta_1} = \frac{E_{2+}}{\eta_2}$$

$$\frac{2E_{1+} - E_{2+}}{\eta_1} - \frac{E_{2+}}{\eta_2} = 0 \Rightarrow \frac{2E_{1+}}{\eta_1} = E_{2+}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$$

$$\frac{E_{2+}}{E_{1+}} = \frac{2}{\eta_1\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)} = \frac{2}{1 + \frac{\eta_1}{\eta_2}}$$

$$\boxed{\tau = \frac{2\eta_2}{\eta_1 + \eta_2}}$$

How are ρ and τ related?

$$1+\rho = \frac{\eta_2+\eta_1}{\eta_2+\eta_1} + \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{2\eta_2}{\eta_2+\eta_1} = \tau$$

Conclusion: $\boxed{1+\rho=\tau}$

Now let's find the impedance at $z=-1$ from the interface (the ratio $\frac{\mathbf{E}_x}{\mathbf{H}_y}$).

Recall: $\mathbf{E}_x = E_+ e^{-jkz} + E_- e^{+jkz}$

$$\mathbf{H}_y = \frac{E_+}{\eta_1} e^{-jkz} - \frac{E_-}{\eta_1} e^{+jkz}$$

The impedance is: $$\frac{\mathbf{E}_x}{\mathbf{H}_y} = \frac{E_+ e^{+jkl} + E_- e^{-jkl}}{\frac{E_+}{\eta_1} e^{+jkl} - \frac{E_-}{\eta_1} e^{-jkl}}$$

$$= \eta_1 \frac{e^{+jkl} + \frac{E_-}{E_+} e^{-jkl}}{e^{+jkl} - \frac{E_-}{E_+} e^{-jkl}}$$

But: $\frac{E_-}{E_+} = \rho = \frac{\eta_2-\eta_1}{\eta_1+\eta_2}$ $\quad 0<\rho<1.$

Then: $$\frac{\mathbf{E}_x}{\mathbf{H}_y} = \eta_1 \frac{e^{+jkl} + \frac{\eta_2-\eta_1}{\eta_1+\eta_2} e^{-jkl}}{e^{+jkl} - \frac{\eta_2-\eta_1}{\eta_1+\eta_2} e^{-jkl}}$$

$$= \eta_1 \frac{(\eta_1+\eta_2)e^{+jkl} + (\eta_2-\eta_1)e^{-jkl}}{(\eta_1+\eta_2)e^{+jkl} - (\eta_2-\eta_1)e^{-jkl}}$$

$$= \eta_1 \frac{\eta_2\left(e^{+jkl}+e^{-jkl}\right) + \eta_1\left(e^{+jkl}-e^{-jkl}\right)}{\eta_1\left(e^{+jkl}+e^{-jkl}\right) + \eta_2\left(e^{+jkl}-e^{-jkl}\right)}$$

or: $$\boxed{\frac{\mathbf{E}_x}{\mathbf{H}_y} = \eta = \eta_1 \frac{\eta_2 \cos kl + j\eta_1 \sin kl}{\eta_1 \cos kl + j\eta_2 \sin kl}}.$$

The concept of field impedance is extremely important and useful.

For nonmagnetic, lossless materials, the reflection coefficient can be written in a form that is quite commonly used:

$$\eta = \left(\frac{\mu_o}{\varepsilon_o \varepsilon}\right)^{1/2} = \eta_o \left(\frac{1}{\varepsilon}\right)^{1/2} = \frac{\eta_o}{n}$$

$$\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{n_1 - n_2}{n_1 + n_2}$$

Example: Glass has a refractive index of about 1.5 for visible wavelengths. Air is very close to 1.

$$\rho = \frac{1.5 - 1}{1.5 + 1} = 0.2 \quad \textit{field reflection}$$

Power reflection $= |\rho|^2 = 0.04$, or about 4%.

Power transmitted $= 96\%$